PHYSICS 3800 Assignment 1

Numerical Differentiation and Integration. 7 questions.

1. (5 pts.) Write the number 43.6875 (base 10) as (i) a binary number in the format $(b_0.b_1b_2b_3b_4\ldots)\times 2^n$, (ii) a IEEE single-precision (32 bit) bitstring. **Show your workings.**

   Beyond our course notes, there is a fine description written by M.L. Overton available at

   http://www.cs.mcgill.ca/~chang/teaching/cs350/doc/overton_ieee.pdf.

   You may check your answer at

   https://www.h-schmidt.net/FloatConverter/IEEE754.html.

2. (5 pts.) Show that Simpson's Rule is exact for any polynomial of degree $n \leq 3$. Use $N = 3$ and integrate from 0 to 1 for convenience.

3. (5 pts.) Compare the exact value of the integral $\int_0^1 x^4 dx$ with the Trapezoid approximation using N=3 and also N=5, and with Simpson's approximation using N=3 points. Use a calculator.

4. (5 pts.) Use Taylor expansions for $f(x+h)$ and $f(x+2h)$ to derive a "one-sided" expression for the derivative $f'(x)$ in terms of $f(x)$, $f(x+h)$ and $f(x+2h)$, where the neglected terms are $O(h^2)$.

5. (5 pts.) Derive the Three-Point formula given in class for the derivative of a function with non-uniform grid spacing:

$$f_i' = \frac{h_{i-1}^2 f_{i+1} + (h_i^2 - h_{i-1}^2) f_i - h_i^2 f_{i-1}}{h_i h_{i-1}(h_i + h_{i-1})} + O(h^2)$$

   where $h_i = x_{i+1} - x_i$. Start by writing Taylor expansions for f_{i+1} and f_{i-1} (see Class Notes).

6. (10 pts.) Verify the Richardson Extrapolation Scheme for $f'(x)$ to order $O(h^6)$:

$$\Delta_1(h) - 20\Delta_1(h/2) + 64\Delta_1(h/4) = 45 f'(x) + O(h^6).$$

   Recall that $\Delta_1(h) = [f(x+h) - f(x-h)]/2h$. Do no use the Taylor series directly, but rather apply the recursion formula shown in class.

7. (10 pts.) Given two polynomials $f(x)$ and $q(x)$, one can always uniquely write $f(x)/q(x) = a(x) + r(x)/q(x)$, where the degree of the remainder $r(x)$ is less than the degree of the divisor $q(x)$.

   (a) Show that the function $f(x) = (1 + x)^5$ can be written as,

   $$f(x) = P_3(x)a_2(x) + r_2(x),$$

   where $P_l(x)$ denotes the l^{th} order Legendre polynomial, by finding the second order polynomials $a_2(x)$ and $r_2(x)$.

   (b) Show explicitly that

   $$f(x_k) = r_2(x_k)$$

   where $\{x_k\}$ denote the roots $P_3(x_k) = 0$. Show explicitly that

   $$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} r_2(x)dx$$

   (c) *Briefly* discuss the significance of this result in the context of the Gaussian quadrature. (See course notes and pages 44 - 47 of the Klein and Godunov handout.)

   NOTE: Although you can do the polynomial division by hand, please feel free to use software like Python for this question. Python also has the capability to do polynomial division through SymPy, with the function sympy.polys.polytools.div (`https://docs.sympy.org/latest/modules/polys/reference.html`). If you have access to it, Mathematica has two commands: `PolynomialQuotient`, which evaluates the quotient $a(x)$, and `PolynomialRemainder`, which evaluates the remainder $r(x)$ for ordinary polynomials with integer exponents and rational-number coefficients for each term. If you do not have access to Mathematica, the website `http://www.wolframalpha.com` should suffice. In your solutions, include the commands you used to get your results if you use a computer algebra system.