Rewrite Taylor series

① $f(x-h) =$

② $f(x+h) =$

subtract: ② − ①

★ $f(x+h) - f(x-h) =$

$f'(x) =$

$f_i' =$

Centred difference or 3-point formula

add: ① + ②

$f_{i+1} + f_i =$

$f_i'' =$

Can generate higher order differences

E.g. $f(x+2h) - f(x-2h) =$

$f'(x) =$

5-point formula

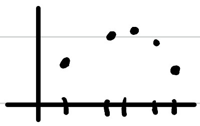
$$f_i' = \frac{1}{12h}\left(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}\right) + \mathcal{O}(h^4)$$

described as $\Big\{$ accurate to order h^4

As we have seen, subtraction reduces precision, so it is advantageous to reduce the number of subtractions

$f_i' =$

Non-uniformly spaced data



$\Delta x_i =$

one option: fit data to analytic function and then take derivative

define $h_i =$

$h_{i-1} =$

Write Taylor series

$f(x_{i\pm1}) =$

or $f_{i+1} =$

$f_{i-1} =$

$f'(x_i) = f_i' =$

3-point formula for non-uniform data

Richardson Extrapolation

Can construct algorithms to get derivatives to arbitrary order

For $f'(x)$: Let $\Delta_1(h) =$

$\Delta_1(h) =$

can show that (5-pt formula)

and

Can determine the coefficients for higher orders systematically

write $\Delta_1(h) =$

call $\Gamma_{k0} =$

can show

$\Gamma_{kl} =$

for $0 \le l \le k$ leads to

$f'(x) =$

where $B_{k\ell} =$

and

$B_{kk} =$

Example: $k = 1, \ell = 1$

$f'(x) =$

$f'(x) =$

$f'(x) =$

Can apply similar method for higher order derivatives

Let $\Delta_2(h) =$

Can show

Error analysis
as h decreases

h

error $\epsilon =$

ϵ is minimized when

ϵ_{ro} from subtraction $\dfrac{f(x+h) - f(x)}{h}$ is $\epsilon_{ro} \sim$

ϵ_{trunc}: forward difference centred diff

minimum error (best you can get) when

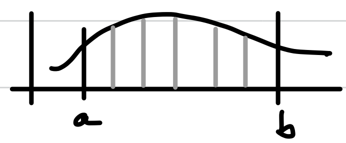
Assume for simplicity

$h_{FD} \simeq$ $h_{CD} \simeq$

$\epsilon_{FD} \simeq$ $\epsilon_{CD} \simeq$

Integration

$$I = \int_a^b f(x)\,dx \longrightarrow$$



Before computers, approximated areas with

Recall Riemann definition / rectangle rule

sum over rectangular areas, take limit as width

For finite h ($h \neq 0$), in general

$$\int_a^b f(x)\,dx \doteq$$

- different algorithms

Trapezoid rule

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- over each interval, area is approximated by area of trapezoid