Gaussian Integration

In Gaussian integration, we fit $f(x)$ over the entire interval $[a,b]$ by a complete set of polynomials

$$f(x) = \sum_{\ell=0}^{n} \alpha'_\ell P_\ell(x)$$

Best to use orthogonal polynomials like the Legendre Polynomials, which satisfy

$$\int_{-1}^{1} dx\, P_n(x) P_m(x) = \frac{2}{2m+1} \delta_{mn} \quad , \quad \delta_{mn} = \begin{cases} 1 & \text{if } m=n \\ 0 & m \neq n \end{cases}$$

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = \frac{1}{2}(3x^2 - 1)$$

defined on $x \in [-1, 1]$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

P_n has n zeros

To change limits of integration, can apply

$$x \to \frac{b-a}{2} x + \frac{b+a}{2}$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} dx\, f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right)$$

Assume $f(x)$ is well approximated by a polynomial of order $2n-1$

$$f(x) \simeq p_{2n-1}(x) \quad \text{(not Legendre polynomial)}$$

$$= c_0 + c_1 x + c_2 x^2 + \dots + c_{2n-1} x^{2n-1}$$

For $$I = \int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{n} w_i f(x_i)$$

we need to find x_i's and w_i's (with polynomial trickery)

Write $$p_{2n-1}(x) = p_{n-1}(x)\, P_n(x) + r_{n-1}(x)$$

Legendre polynomial

two different polynomials of order $n-1$ (different coefficients)

e.g. $$p_3(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 = p_1 P_2 + r_1$$

$$= (d_0 + d_1 x)\left[\tfrac{1}{2}(3x^2-1)\right] + (e_0 + e_1 x)$$

with
$$c_0 = -\tfrac{1}{2} d_0 + e_0$$
$$c_1 = -\tfrac{1}{2} d_1 + e_1$$
$$c_2 = \tfrac{3}{2} d_0$$
$$c_3 = \tfrac{3}{2} d_1$$

$$p_{2n-1} = p_{n-1} P_n + r_{n-1}$$

express $p_{n-1} = \sum_{k=0}^{n-1} \alpha_k P_k$ (in basis of Legendre polynomials)

$$I = \int_{-1}^{1} dx\, f(x) \simeq \int_{-1}^{1} dx\, p_{2n-1}(x) = \int_{-1}^{1} \left[\left(\sum_{k=0}^{n-1} \alpha_k P_k \right) P_n + r_{n-1} \right] dx$$

$$= \sum_{k=0}^{n-1} \alpha_k \underbrace{\int_{-1}^{1} dx\, P_k(x) P_n(x)}_{\therefore\ = 0} + \int_{-1}^{1} dx\, r_{n-1}(x)$$

$k \neq n$ because k goes from 0 to $n-1$

$$\therefore \quad I = \int_{-1}^{1} dx\, r_{n-1}(x)$$

Let the n zeros of $P_n(x)$ be $x_1, x_2, \ldots x_n$ (these are known)

Then, at these zeros,

$$p_{2n-1}(x_i) = p_{n-1}(x_i) \underbrace{P_n(x_i)}_{0} + r_{n-1}(x_i)$$

$$p_{2n-1}(x_i) = r_{n-1}(x_i)$$

Now express $r_{n-1}(x) = \sum_{k=0}^{n-1} \tilde{\omega}_k P_k(x)$ (in basis of Legendre polynomials)

so $p_{2n-1}(x_i) = r_{n-1}(x_i) = \sum_{k=0}^{n-1} \tilde{\omega}_k P_k(x_i)$

where x_i is one of the n roots of $P_n(x)$
($P_n(x_i) = 0$, $i = 1, \dots n$)

e.g. $n = 3$ ($2n-1 = 5$, $k_{max} = n-1 = 2$)

$$p_{2n-1}(x_1) = p_5(x_1) = \tilde{\omega}_0 P_0(x_1) + \tilde{\omega}_1 P_1(x_1) + \tilde{\omega}_2 P_2(x_1)$$

$$p_5(x_2) = \tilde{\omega}_0 P_0(x_2) + \tilde{\omega}_1 P_1(x_2) + \tilde{\omega}_2 P_2(x_2)$$

$$p_5(x_3) = \tilde{\omega}_0 P_0(x_3) + \tilde{\omega}_1 P_1(x_3) + \tilde{\omega}_2 P_2(x_3)$$

3 eq$^{\underline{n}}$s, 3 unknowns – $\tilde{\omega}_0$, $\tilde{\omega}_1$ and $\tilde{\omega}_2$

or in general

$$\begin{pmatrix} p_{2n-1}(x_1) \\ p_{2n-1}(x_2) \\ \vdots \\ p_{2n-1}(x_n) \end{pmatrix} = \begin{pmatrix} P_0(x_1) & \dots & P_{n-1}(x_1) \\ P_0(x_2) & \dots & P_{n-1}(x_2) \\ \vdots & & \vdots \\ P_0(x_n) & \dots & P_{n-1}(x_n) \end{pmatrix} \begin{pmatrix} \tilde{\omega}_0 \\ \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_{n-1} \end{pmatrix}$$

$n \times n$ matrix, $\overleftrightarrow{P}$

and its inverse, $\overleftrightarrow{P}^{-1}$

Then

$$\begin{pmatrix} \tilde{\omega}_0 \\ \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_n \end{pmatrix} = \overleftrightarrow{P}^{-1} \begin{pmatrix} p_{2n-1}(x_1) \\ p_{2n-1}(x_2) \\ \vdots \\ p_{2n-1}(x_n) \end{pmatrix}$$

or $$\tilde{\omega}_k = \sum_{i=1}^{n} p_{2n-1}(x_i) \left\{ \overleftrightarrow{P}^{-1} \right\}_{ki}$$

back to integral

$$I = \int_{-1}^{1} f(x)\,dx \doteq \int_{-1}^{1} p_{2n-1}(x)\,dx = \int_{-1}^{1} r_{n-1}(x)\,dx = \int_{-1}^{1} \sum_{k=0}^{n-1} \tilde{\omega}_k P_k(x)\,dx$$

multiply by $1 = P_0(x)$

$$I = \sum_{k=0}^{n-1} \tilde{\omega}_k \int_{-1}^{1} P_k(x) P_0(x)\,dx = \sum_{k=0}^{} \tilde{\omega}_k \frac{2}{2k+1} \delta_{k0} = 2\tilde{\omega}_0$$

$$I = 2 \sum_{i=1}^{n} p_{2n-1}(x_i) \left\{ \overleftrightarrow{P}^{-1} \right\}_{0i}$$

but $p_{2n-1}(x) \doteq f(x)$

and calling $\omega_i = 2 \left\{ \overleftrightarrow{P}^{-1} \right\}_{0i}$

we get

$$I = \sum_{i=1}^{n} f(x_i)\, \omega_i$$

The ω_i's are the weights for $f(x)$ evaluated at the zeros $\{x_i\}$ of $P_n(x)$.

see Mathematica example, K and G handout