Partial Differential Equations - PDEs

Three main types of PDEs in physics

1) Elliptic - static problems (∇^2, no time)

   e.g.

   - generally, boundary conditions given around

2) Parabolic (∇^2 and $\partial/\partial t$)

   e.g.

3) Hyperbolic - propagation (∇^2 and $\partial^2/\partial t^2$, but not always)

   e.g.

Note: Schrödinger's eq$^{\underline{n}}$

can be viewed as a diffusion eq$^{\underline{n}}$ with

For a general form

$$a \frac{\partial^2 V}{\partial x^2} + b \frac{\partial^2 V}{\partial x \partial y} + c \frac{\partial^2 V}{\partial y^2} + d \frac{\partial V}{\partial x} + e \frac{\partial V}{\partial y} + fV + g = 0$$

These are examples of - linear in dependent variable

Another important eq$^{\underline{n}}$ is the applied to a small volume of fluid

(for incompressible fluid)

$\vec{v}$ - η -

ρ - P -

- the (diffusive) term gives a character.
If this term is negligible, the eq$^{\underline{n}}$ is more .

Also important in fluids is the

The different types of PDEs have different

We will consider the different approaches in the next lectures.

Do you need a numerical solver?

Separation of Variables

Simplifying PDE analytically can make applying numerical algorithms easier.

E.g. 1-D Wave eq$^{\underline{n}}$ $\quad \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$

string with ends fixed at $x = 0$ and $x = L$

c - wave speed $\sqrt{\dfrac{T}{\rho}}$, T - tension, $\rho = \dfrac{M}{L}$

B.C.'s $\quad u(x=0, t) = 0 = u(x=L, t)$

assume $u(x,t) = y(x) f(t)$ - doesn't always work

plug into PDE ...

$\rightarrow y(x) = A \sin(kx) + B \cos(kx)$ $\quad$ B.C. $\rightarrow B = 0$,

$k = k_n = \dfrac{n\pi}{L}$, $n = 1, 2, \ldots$

$f(t) = C \sin(\omega_n t) + D \cos(\omega_n t)$ $\quad \omega_n^2 = c^2 k_n^2$

...

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \right) \sin(k_n x)$$

$$a_n = \frac{2}{\omega_n L} \int_0^L v_0(x) \sin(k_n x)\, dx$$

$$b_n = \frac{2}{L} \int_0^L u_0(x) \sin(k_n x)\, dx$$

$u_0(x)$ initial displacement, $v_0(x)$ initial velocity

Problem of solving PDE reduces to evaluating definite integrals numerically, unless $u_0(x)$ and $v_0(x)$ are simple.

Otherwise, if eq$^{\underline{n}}$ not separable, must discretize PDE.

Example: Time-Dependent Schrödinger Equation

$$H\psi(\vec{r},t) = i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t)$$

- for simplicity, set $m =$, $\hbar =$

1-D

- parabolic eq$^{\underline{n}}$ - ψ - $|\psi|^2$

B.C.

normalization

In discretized system: $\psi(x,t) \rightarrow$ n-

j-

- magnitude

$\therefore$ want an algorithm

equivalent to

$$H = -\partial^2/\partial x^2 + V(x)$$

Formal sol$^{\underline{n}}$ to $i\frac{\partial}{\partial t}\psi = H\psi$ is

For $t \rightarrow \Delta t$, approximate

So might use

but is not unitary, so probability won't be conserved.

Instead, use for finite difference

expression

$$e^{-iH\Delta t} \simeq$$

which is

Aside

$P =$, $P^{+} =$

$P P^{+} =$

$\tilde{P} \tilde{P}^{+} =$

H Hermitian

$H = H^{+}$

so we get $\psi_j^{n+1} =$

or

$H \psi_j^{n+1} =$

$(1 + \frac{1}{2} H i \Delta t) \psi_j^{n+1} =$

Similarly $\left(1 - \tfrac{1}{2} H i \Delta t\right) \psi_j^n$

$=$

call

$T = \left(\qquad \right)$

$T^* = \left(\qquad \right)$

get

or $$T \vec{\psi}^{\,n+1} = \vec{b}$$

$\uparrow$ $\qquad\qquad\qquad$ $\vec{b} =$

solve for unknown
wavefunction at next
time step

$$\vec{\psi} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Boundary conditions imply