Discrete Solutions of Hyperbolic Equations (Wave eq$\underline{n}$)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \Big(\qquad\qquad\qquad \Big) u(x,t) = 0$$

$$\rightarrow \Big(\qquad\qquad\qquad \Big)\Big(\qquad\qquad\qquad \Big) u(x,t) = 0$$

write as two eq$\underline{n}$s : (1)

(2)

Note: If $u(x,t=0) =$

and $\left.\frac{\partial u}{\partial t}\right|_{t=0} =$, taking eq$\underline{n}$ (2) at $t=0$

gives $z_0(x) =$

Eq$\underline{n}$ (1) is independent of u

This yields

Eq$\underline{n}$ (1) is similar to continuity eq$\underline{n}$ (L16) and is called the

Try simple algorithm:

Euler for time

and

centred diff for space

get $\dfrac{z_j^{n+1} - z_j^n}{\Delta t} + c\left(\dfrac{z_{j+1}^n - z_{j-1}^n}{2\Delta x}\right) = 0$

$$z_j^{n+1} =$$

However, this method is
(We shall a little later examine .)

<u>Lax Method</u> - slight modification of the above

Replace first term on RHS of algorithm with a

$z_j^n \rightarrow$ to get

$$z_j^{n+1} =$$

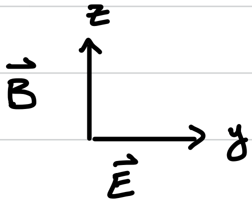
Analysis shows that this approach is stable for all wavelengths if

Example Maxwell's Equations

$\nabla \times \vec{E} =$

in free space
- no currents

$\nabla \times \vec{B} =$



Let $\vec{E} =$, $\vec{B} =$

$\rightarrow \nabla \times \vec{E} =$ $\nabla \times \vec{B} =$

Other discretization schemes for hyperbolic eq$^{\underline{n}}$s include
- Lax-Wendroff methods
- Leap-frog
- Lelevier

} Numerical Recipes

Two-step Lax-Wendroff

half-steps are first calculated with

$z_{j+1/2}^{n+1/2} =$

$z_{j-1/2}^{n+1/2} =$

then find $z_j^{n+1} =$

More generally these steps can be written for

$$\frac{\partial u}{\partial t} =$$

$$u_{j+1/2}^{n+1/2} =$$

$$u_j^{n+1} =$$

F is a function of

Wave eq$^{\underline{n}}$ can be written in this form too

via

$$\frac{\partial r}{\partial t} = \qquad \text{and} \qquad \frac{\partial s}{\partial t} =$$

where $r \equiv$ and $s \equiv$

Calling $\vec{v} =$ $\quad \vec{F}(\vec{v}) = \Big(\qquad \Big) \cdot \vec{\ } =$

Wave eq$^{\underline{n}}$ becomes

note:

Once s is obtained, original f$^{\underline{n}}$ u can be determined via

$$u_j^{n+1} =$$

Discrete Parabolic Equations: Diffusion

1-D

simple approach: Euler for t:

Centred diff. for x:

$u_j^{n+1} =$

stability condition is Δt

Drawback: If we need to improve accuracy by

then we must use

-> # of computations

Dufort - Frankel Method

Trick: Replace

and take

$u_j^{n+1} =$

call $\alpha =$ and solve for

$u_j^{n+1} =$

This algorithm is

While accuracy demands sufficiently small ,

there is no longer a relation between them that must

be satisfied

Since u_j^{n+1} appears on

, the Dufort-Frankel method is an

example of an method,

which are often

Recall trapezoid rule for integration

$$\int_{t_i}^{t_f} f(t)\,dt =$$

For an ODE this translates to

or $y_{i+1} =$

for decay and $|g|$

oscillation , $|g|^2 =$

for growth, call error $\delta y_i =$, ϵ is

and we want (absolute error grows since

)

[unstable: solution becomes large when]

$$\delta y_{i+1} = g\,\delta y_i$$

$$y_{i+1}\,\epsilon_{i+1} =$$

→ stable also for growth
(relative error is stable)

for Diffusion-like equations

$$\frac{\partial u}{\partial t} =$$

other implicit methods include

Backwards Euler and Crank-Nicolson

need to solve algebraic equations for

E.g. Backward Euler for

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} =$$

or

or

$$A \vec{u}^{n+1} = \vec{b} =$$

- matrix eq$^{\underline{n}}$ can be solved efficiently