Discrete Solutions of Hyperbolic Equations (Wave eq$\underline{n}$)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0$$

$$\rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \right) u(x,t) = 0$$

write as two eq$\underline{n}$s: (1) $\frac{\partial z}{\partial t} + c \frac{\partial}{\partial x} z = 0$

(2) $\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = z$

Note: If $u(x, t=0) = u_0$

and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0$, taking eq$\underline{n}$ (2) at $t=0$

gives $z_0(x) = v_0 - c \frac{\partial}{\partial x} u_0$

Eq$\underline{n}$ (1) is independent of u and can be solved on its own.
This yields $z(x,t) \rightarrow$ plug into (2) and solve for u.

Eq$\underline{n}$ (1) is similar to continuity eq$\underline{n}$ (L16) and is called the Advection equation.

Try simple algorithm:

Euler for time $\frac{\partial z}{\partial t} = \frac{z^{n+1} - z^n}{\Delta t} + \mathcal{O}(\Delta t)$

and

centred diff for space

$$\frac{\partial z}{\partial x} = \frac{z_{j+1} - z_{j-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

get $$\frac{z_j^{n+1} - z_j^n}{\Delta t} + c\left(\frac{z_{j+1}^n - z_{j-1}^n}{2\Delta x}\right) = 0$$

time index

$$z_j^{n+1} = z_j^n - \frac{c\,\Delta t}{2\Delta x}\left(z_{j+1}^n - z_{j-1}^n\right)$$

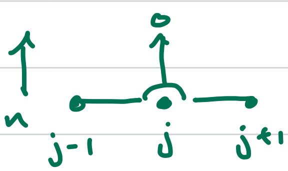
space index

However, this method is unstable for any Δt and Δx.
(We shall a little later examine stability.)

<u>Lax Method</u> - slight modification of the above

Replace first term on RHS of algorithm with a spatial average

$$z_j^n \rightarrow \tfrac{1}{2}\left(z_{j+1}^n + z_{j-1}^n\right)$$ to get



$$z_j^{n+1} = \frac{1}{2}\left(z_{j+1}^n + z_{j-1}^n\right) - \frac{c\,\Delta t}{2\Delta x}\left(z_{j+1}^n - z_{j-1}^n\right)$$

Analysis shows that this approach is stable for all wavelengths if

$$\frac{\Delta x}{\Delta t} \geq c$$

(Courant-Friedrichs-Lewy condition)

wave speed

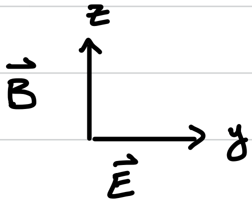
speed of information propagation in algorithm

Example Maxwell's Equations

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

in free space
- no currents

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$



Let $\vec{E} = E(x,t)\,\hat{y}$, $\vec{B} = B(x,t)\,\hat{z}$

$\rightarrow \nabla \times \vec{E} = \frac{\partial E}{\partial x}\hat{z}$ $\qquad \nabla \times \vec{B} = -\frac{\partial B}{\partial x}\hat{y}$

$$\frac{\partial B}{\partial t} + c\frac{\partial E}{\partial x} = 0 \qquad B_j^{n+1} = \frac{1}{2}\left(B_{j+1}^n + B_{j-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(E_{j+1}^n - E_{j-1}^n\right)$$

$$\frac{\partial E}{\partial t} + c\frac{\partial B}{\partial x} = 0 \qquad E_j^{n+1} = \frac{1}{2}\left(E_{j+1}^n + E_{j-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(B_{j+1}^n - B_{j-1}^n\right)$$

=

Other discretization schemes for hyperbolic eq$^{\underline{n}}$s include
- Lax-Wendroff methods
- Leap-frog
- Lelevier

$\}$ Numerical Recipes

Two-step Lax-Wendroff $\qquad \frac{\partial z}{\partial t} + c\frac{\partial z}{\partial x} = 0$

half-steps are first calculated with regular Lax scheme

$$z_{j+1/2}^{n+1/2} = \frac{1}{2}\left(z_{j+1}^n + z_j^n\right) - \frac{c\Delta t}{2\Delta x}\left(z_{j+1}^n - z_j^n\right)$$

$$z_{j-1/2}^{n+1/2} = \frac{1}{2}\left(z_j^n + z_{j-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(z_j^n - z_{j-1}^n\right)$$

then find $z_j^{n+1} = z_j^n - c\frac{\Delta t}{\Delta x}\left(z_{j+1/2}^{n+1/2} - z_{j-1/2}^{n+1/2} \right)$

More generally these steps can be written for

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} F(u) \qquad F \text{ is conserved flux}$$

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_{j+1}^n + u_j^n \right) - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_j^n \right)$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2} \right)$$

$$u_{j-1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j-1}^n \right) - \frac{\Delta t}{2\Delta x}\left(F_j^n - F_{j-1}^n \right)$$

F is a function of u and its spatial derivatives.

Wave eq$^{\underline{n}}$ $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ can be written in this form too

via

$$\frac{\partial r}{\partial t} = c\frac{\partial s}{\partial x} \qquad \text{and} \qquad \frac{\partial s}{\partial t} = c\frac{\partial r}{\partial x}$$

where $r \equiv c\frac{\partial u}{\partial x}$ and $s \equiv \frac{\partial u}{\partial t}$

Calling $\vec{v} = \begin{pmatrix} r \\ s \end{pmatrix}$ $\quad \vec{F}(\vec{v}) = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \cdot \vec{v} = \begin{pmatrix} -cs \\ -cr \end{pmatrix}$

Wave eq$^{\underline{n}}$ becomes $\frac{\partial \vec{v}}{\partial t} = -\frac{\partial \vec{F}}{\partial x}$

note:

Once s is obtained, original f$^{\underline{n}}$ u can be determined via

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2}\left(s_j^{n+1} + s_j^n \right)$$

Discrete Parabolic Equations: Diffusion

1-D $\quad \dfrac{\partial u}{\partial t} - D \dfrac{\partial^2 u}{\partial x^2} = 0$

simple approach: Euler for t: $\quad \dfrac{\partial u}{\partial t} \doteq \dfrac{u^{n+1} - u^n}{\Delta t}$

Centred diff. for x:

$$u_j^{n+1} = u_j^n + D \frac{\Delta t}{(\Delta x)^2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right)$$

stability condition is $\quad \Delta t \leq \dfrac{1}{2} \dfrac{(\Delta x^2)}{D}$

Drawback: If we need to improve accuracy by $\Delta x \to \Delta x/2$

then we must use $\Delta t \to \Delta t/4$

$\to$ # of computations increases by factor of 8

Dufort-Frankel Method

Trick: Replace $2u_j^n$ on RHS by time average $u_j^{n+1} + u_j^{n-1}$

and take a double time step

$$u_j^{n+1} = u_j^{n-1} + \frac{2D\Delta t}{\Delta x^2} \left(u_{j+1}^n - (u_j^{n+1} + u_j^{n-1}) + u_{j-1}^n \right)$$

call $\alpha = \dfrac{2D\Delta t}{\Delta x^2}$ and solve for u_j^{n+1}

$$u_j^{n+1} = \left(\frac{1-\alpha}{1+\alpha} \right) u_j^{n-1} + \frac{\alpha}{1+\alpha} \left(u_{j+1}^n + u_{j-1}^n \right)$$

This algorithm is unconditionally stable.
While accuracy demands sufficiently small Δt and Δx, there is no longer a relation between them that must be satisfied to maintain stability.

Since u_j^{n+1} appears on both sides of the finite difference equation, the Dufort-Frankel method is an example of an intrinsic (also called implicit) method, which are often unconditionally stable.

Recall trapezoid rule for integration

$$\int_{t_i}^{t_f} f(t)\,dt = \frac{\Delta t}{2}\left(f_{i+1} + f_i\right)$$

For an ODE $y' = \alpha y$ this translates to

$$y_{i+1} - y_i = \frac{\Delta t}{2}\alpha\left(y_{i+1} + y_i\right)$$

or

$$y_{i+1} = \left(\frac{1 + \frac{1}{2}\Delta t\alpha}{1 - \frac{1}{2}\Delta t\alpha}\right) y_i$$

This is g from our ODE stability analysis.

for decay $\alpha < 0$ and $|g| < 1$

oscillation $\alpha = i\omega$, $|g|^2 = \frac{1 + \frac{1}{2}\Delta t\, i\omega}{1 - \frac{1}{2}\Delta t\, i\omega} \cdot \frac{1 - \frac{1}{2}\Delta t\, i\omega}{1 + \frac{1}{2}\Delta t\, i\omega} = 1$

for growth, call error $\delta y_i = y_i \epsilon_i$, ϵ is relative error

and we want ϵ not to grow (absolute error grows since solution grows exponentially)

[unstable: solution becomes large when true solution does not]

$$\delta y_{i+1} = g\,\delta y_i$$

$$y_{i+1}\epsilon_{i+1} = g\, y_i \epsilon_i$$

$$g\, y_i\, \epsilon_{i+1} = g\, y_i\, \epsilon_i \quad \rightarrow \text{ stable also for growth}$$

$$\epsilon_{i+1} = \epsilon_i \qquad \text{(relative error is stable)}$$

for Diffusion-like equations

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$$

other implicit methods include

Backwards Euler and Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^{n+1} \qquad\qquad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(F_i^{n+1} + F_i^n\right)$$

need to solve algebraic equations for u_i^{n+1}

E.g. Backward Euler for $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$ $\Big\}$ $F = D\frac{\partial^2 u}{\partial x^2}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^{n+1} = \frac{D}{\Delta x^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)$$

or

$$-u_{i-1}^{n+1} + u_i^{n+1}\left(2 + \frac{\Delta x^2}{D\Delta t}\right) - u_{i+1}^{n+1} = D\frac{\Delta t}{\Delta x^2} u_i^n$$

or

$$A\vec{u}^{n+1} = \vec{b} = \frac{D\Delta t}{\Delta x^2}\vec{u}^n$$

tridiagonal

- matrix eq$^{\underline{n}}$ can be solved efficiently