von Neumann Stability Analysis

Recall solution to the wave eq$^{\underline{n}}$ $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ by Sep. of variables

$u(x,t) =$

We've seen that $\frac{\partial^2 y}{\partial x^2} \rightarrow$ with A

So the separated eq$^{\underline{n}}$s are
and the

The sol$^{\underline{n}}$ is of the form $\sum_m (a_m \sin \omega_m t + b_m \cos \omega_m t) \sin k_m x$

so consider a contribution to the discretized solution

$u_j^n =$

or generally $u_j^n =$

eigenmode contribution to sol$^{\underline{n}}$ with $k = \frac{2\pi}{\lambda}$
How does it grow in time?

Increasing time means

Solution is unstable if

Recall "simple" algorithm for advection eq^{n}:

$u_j^{n+1} =$

sub in $u_j^{n} =$

$\xi \quad e \quad =$

$\xi =$

$\Rightarrow |\xi|$

For Lax Method

$u_j^{n+1} =$

Elliptic Equations - PDEs in matrix form

We've already seen 1-D Poisson eq^n. Now 2-D

discretize: $x_j =$

$y_l =$

using $\frac{\partial^2}{\partial x^2} \rightarrow$ gives

ASIDE: Can rearrange eq^n

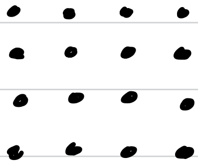
$u_{j,l} =$

i.e. $u_{j,l}$ is the average value of its neighbours plus a direct contribution from ρ. This forms the basis for relaxation methods, the simplest of which plugs in old values of $u_{j,l}$ in the RHS to generate new values - iterating until results converge (Jacobi method).

Want to make a single (not 2-D matrix).

call $i =$ for $j = 0, 1, \ldots J$ and $l = 0, 1, \ldots L$

e.g. $J = L = 3$ (4×4)



$j = 0 \rightarrow i =$ (column 1)

$j = \quad \rightarrow i =$

$j = \quad \rightarrow i =$

$j = \quad \rightarrow i =$

$(j, l) = (0, 0)$

$i =$

for $j \rightarrow j+1$, $i \rightarrow$

$j \rightarrow$

$l \rightarrow$

$l \rightarrow$

Difference eq$^{\underline{n}}$ $u_{j+1,l} + u_{j-1,l} + u_{j,l+1} + u_{j,l-1} - 4u_{j,l} = h^2 \rho_{j,l}$

becomes

4 neighbours of

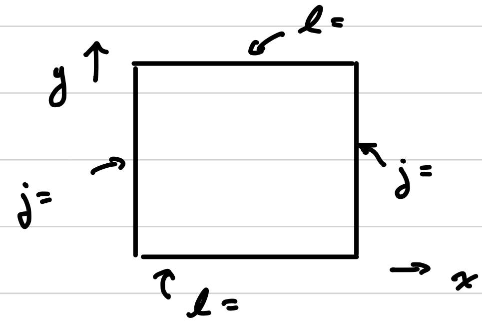
holds only at

$j =$ $l =$

At boundary points:

$j =$ $\rightarrow$ $i =$

$j =$ $\rightarrow$ $i =$



$\ell =$ $\rightarrow$ $i =$

$\ell =$

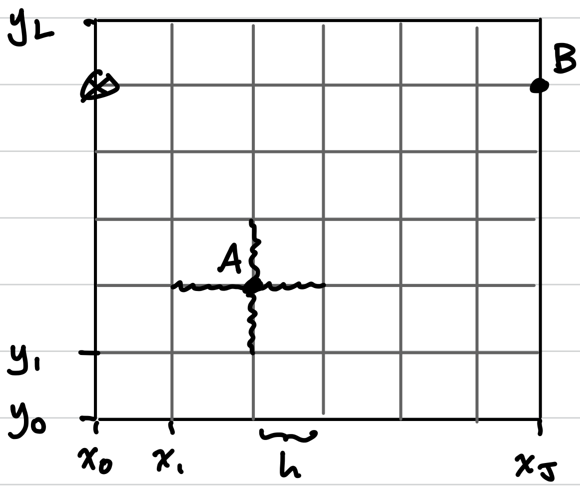
must specify either u or its derivative

E.g. If, say, $\left.\frac{\partial u}{\partial x}\right|_{x_J, y} = 0$

$\rightarrow$ $= 0$ $\rightarrow$

or $u_{i_{rightboundary}} = u_{i_{rightboundary}} - (\quad)$

with $i_{rightboundary}$ given above for $j = J$ case



2^{nd} derivative at an interior point A is evaluated using 4 nearest neighbours $(j \pm 1, \ell \pm 1)$ and itself (j, ℓ)

- Boundary points B must be specified

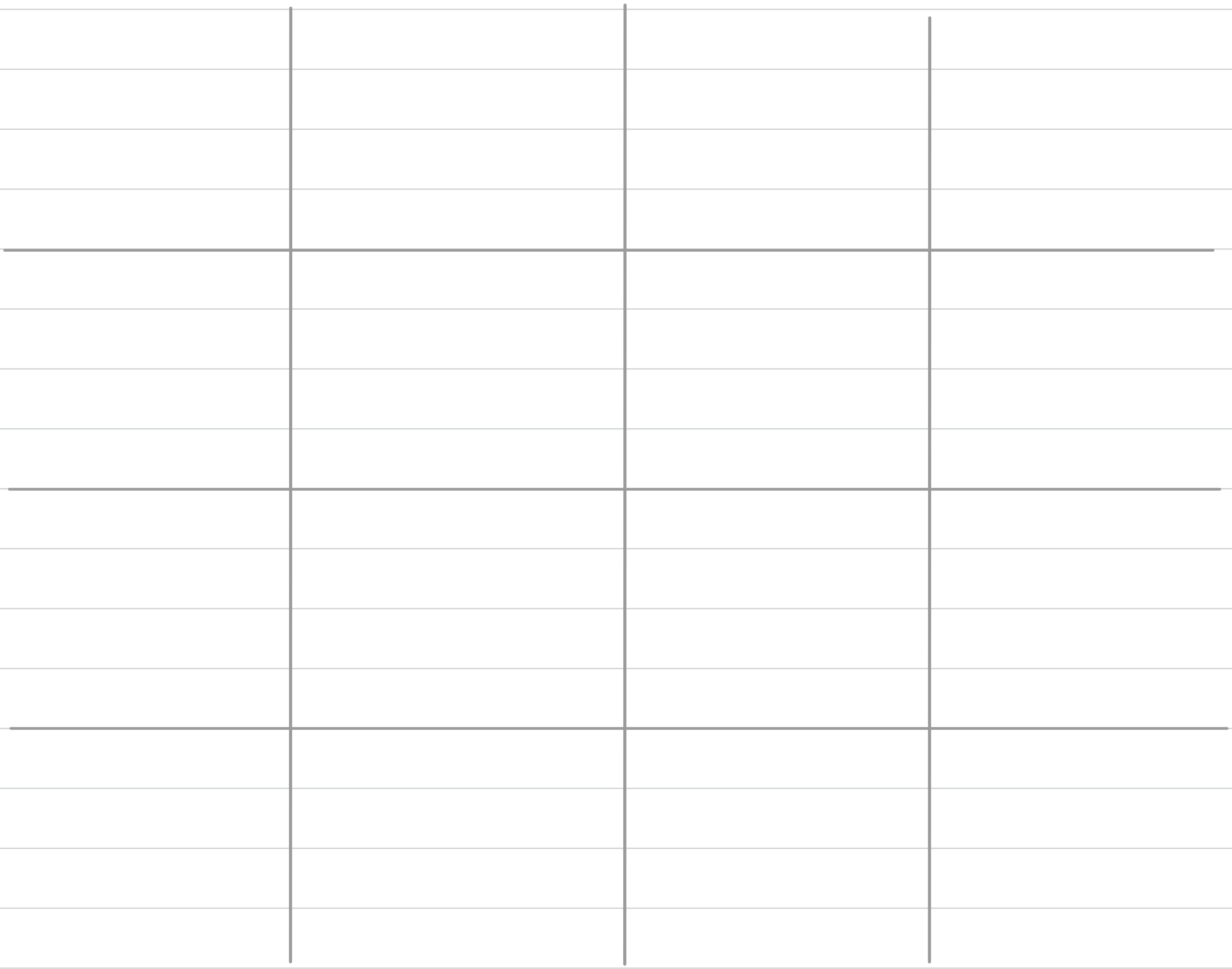
or

- With periodic boundary conditions 2^{nd} derivative at B involves point $\otimes$

Solving the PDE reduces to solving

where A is

a.k.a.



Each sub-block is

There are sub-blocks → A is

Use linear algebra routines for band matrices

Names of relaxation methods include

- Gauss-Seidel
- Successive Over-Relaxation