von Neumann Stability Analysis

Recall solution to the wave eq^{n} $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ by Sep. of variables

$u(x,t) = y(x) f(t) \rightarrow \ddot{f} = -\omega_n^2 f$

$y'' = -k_n^2 y$

We've seen that $\frac{\partial^2 y}{\partial x^2} \rightarrow A\vec{y}$ with A tridiagonal on discretization

So the separated eq^{n}s are eigenvalue eq^{n}s
and the k's are eigenvalues

The sol^{n} is of the form $\sum_m (a_m \sin \omega_m t + b_m \cos \omega_m t) \sin k_m x$

So consider a contribution to the discretized solution

$u_j^n \sim \cos(\omega_m n \Delta t) \sin(k_m j \Delta x) \sim (e^{\omega_m \Delta t})^n \sin(k_m j \Delta x)$

or generally $u_j^n = \xi^n e^{ikj\Delta x}$

eigenmode contribution to sol^{n} with $k = \frac{2\pi}{\lambda}$
How does it grow in time?

Increasing time means increasing powers of ξ

Solution is unstable if $|\xi| > 1$ for some k

Recall "simple" algorithm for advection eq^{n}: $\dfrac{\partial u}{\partial t} = -c\,\dfrac{\partial u}{\partial x}$

$\uparrow$ Euler $\qquad\qquad \uparrow$ centred

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

sub in $u_j^n = \xi^n e^{ik\Delta x j}$

$$\xi^{n+1} e^{ik\Delta x j} = \xi^n e^{ik\Delta x j} - \frac{i}{i}\frac{c\Delta t}{2\Delta x}\xi^n\left(e^{ik\Delta x j}e^{ik\Delta x} - e^{ik\Delta x j}e^{-ik\Delta x}\right)$$

$$\xi = 1 - i\frac{c\Delta t}{\Delta x}\sin(k\Delta x)$$

$\Rightarrow |\xi| > 1$ for all k $\therefore$ unstable

For Lax Method

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$\xi^{n+1}e^{ik\Delta x j} = \frac{1}{2}\xi^n e^{ik\Delta x j}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - \frac{i}{i}\frac{c\Delta t}{2\Delta x}\xi^n e^{ik\Delta x j}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$\xi = \cos(k\Delta x) - i\frac{c\Delta t}{\Delta x}\sin(k\Delta x)$$

$$|\xi|^2 = \cos^2(k\Delta x) + \left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x)$$

$|\xi|^2 < 1$ if $\left(\dfrac{c\Delta t}{\Delta x}\right)^2 < 1$ or $\dfrac{\Delta x}{\Delta t} > c$

Elliptic Equations - PDEs in matrix form

We've already seen 1-D Poisson eq^n. Now 2-D

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y)$$

discretize: $x_j = x_0 + jh$, $j = 0, 1, \dots J$
$y_\ell = y_0 + \ell h$, $\ell = 0, 1, \dots L$

using $\frac{\partial^2}{\partial x^2} \to \frac{1}{h^2}\left(u_{j+1,\ell} - 2u_{j,\ell} + u_{j-1,\ell}\right)$ gives

$$u_{j+1,\ell} + u_{j-1,\ell} + u_{j,\ell+1} + u_{j,\ell-1} - 4u_{j,\ell} = h^2 \rho_{j,\ell}$$

ASIDE: Can rearrange eq^n

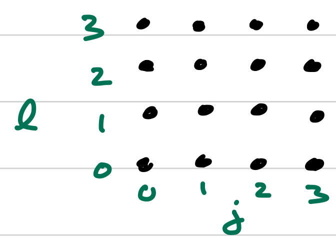
$$u_{j,\ell} = \frac{1}{4}\left(u_{j+1,\ell} + u_{j-1,\ell} + u_{j,\ell+1} + u_{j,\ell-1}\right) - \frac{h^2}{4}\rho_{j,\ell}$$

i.e. $u_{j,\ell}$ is the average value of its neighbours plus a direct contribution from ρ. This forms the basis for relaxation methods, the simplest of which plugs in old values of $u_{j,\ell}$ in the RHS to generate new values - iterating until results converge (Jacobi method).

Want to make $u_{j,\ell}$ a single 1-D vector (not 2-D matrix).

call $i = j(L+1) + \ell$ for $j = 0, 1, \dots J$ and
$\ell = 0, 1, \dots L$

e.g. $J = L = 3$ (4×4)



16 points

4 columns

$j=0$	$\rightarrow$ $i = \ell$	$0, 1, 2, 3$	(column 1)
$j=1$	$\rightarrow$ $i = 4+\ell$	$4, 5, 6, 7$	column 2
$j=2$	$\rightarrow$ $i = 8+\ell$	$8, 9, 10, 11$	col 3
$j=3$	$\rightarrow$ $i = 12+\ell$	$12, 13, 14, 15$	col 4

(j,ℓ) =	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$	$(1,1)$	$(1,2)$	...	$(3,2)$	$(3,3)$
i =	0	1	2	3	4	5	6		14	15

for $j \rightarrow j+1$, $\quad i \rightarrow i + (L+1)$

$j \rightarrow j-1$, $\quad i \rightarrow i - (L+1)$

$\ell \rightarrow \ell+1$ $\quad i \rightarrow i+1$

$\ell \rightarrow \ell-1$ $\quad i \rightarrow i-1$

Difference eq$^{\underline{n}}$ $\quad u_{j+1,\ell} + u_{j-1,\ell} + u_{j,\ell+1} + u_{j,\ell-1} - 4u_{j,\ell} = h^2 \rho_{j,\ell}$

becomes

$$u_{i+L+1} + u_{i-(L+1)} + u_{i+1} + u_{i-1} - 4u_i = h^2 \rho_{j,\ell}$$

4 neighbours of u_i

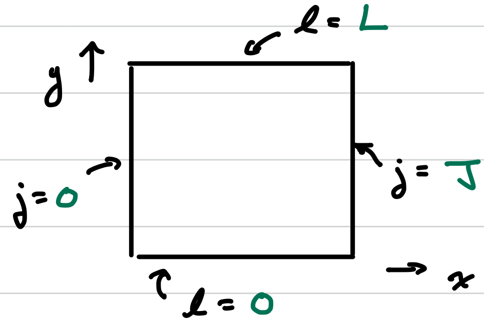
holds only at

$j = 1, 2, \ldots J-1 \qquad \ell = 1, 2, \ldots L-1$

At boundary points :

$j = 0 \rightarrow i = 0, \dots L$

$j = J \rightarrow i = J(L+1), J(L+1)+1,$
$J(L+1)+2, \dots J(L+1)+L$



$\ell = 0 \rightarrow i = 0, L+1, 2(L+1), \dots J(L+1)$

$\ell = L \rightarrow i = L, (L+1)+L, 2(L+1)+L, \dots J(L+1)+L$

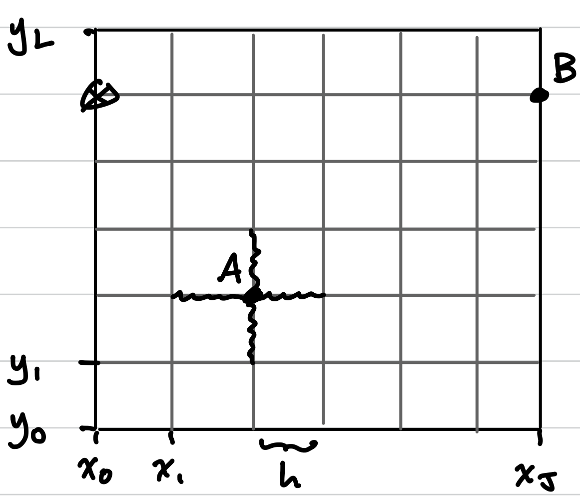
must specify either u or its derivative

E.g. If, say, $\left.\frac{\partial u}{\partial x}\right|_{x_J, y} = 0$

$$\rightarrow \frac{u_{J,\ell} - u_{J-1,\ell}}{h} = 0 \rightarrow u_{J,\ell} = u_{J-1,\ell}$$

or $u_{i_{rightboundary}} = u_{i_{rightboundary}-(L+1)}$

with $i_{rightboundary}$ given above for $j = J$ case



2^{nd} derivative at an interior point A is evaluated using 4 nearest neighbours $(j \pm 1, \ell \pm 1)$ and itself (j, ℓ)

- Boundary points B must be specified

or

- With periodic boundary conditions 2^{nd} derivative at B involves point ⊗

Solving the PDE reduces to solving

$A\vec{u} = \vec{b}$ where A is tridiagonal with fringes

a.k.a. a band matrix

$$
\left[\begin{array}{c|c|c|c}
\begin{matrix} -4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -4 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -4 & 1 & 0 & \\ 0 & 0 & 1 & -4 & 1 & \cdots \\ \vdots & & & \ddots & 1 & -4 \end{matrix}
&
\begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & & & & 1 \end{matrix}
& 0 & 0 \\ \hline
\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{matrix}
&
\begin{matrix} -4 & 1 & 0 & 0 & \\ 1 & -4 & 1 & 0 & \\ 0 & 1 & -4 & 1 & \\ & & & \ddots & \\ & & & -4 & 1 \\ & & & 1 & -4 \end{matrix}
&
\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & \ddots & \\ & & 1 & 0 \\ & & 0 & 1 \end{matrix}
& 0 \\ \hline
0
&
\begin{matrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & \ddots & \\ & & 1 & 0 \\ & & 0 & 1 \end{matrix}
&
\begin{matrix} -4 & 1 & 0 & 0 & \\ 1 & -4 & 1 & 0 & \cdots \\ 0 & 1 & -4 & 1 & \\ 0 & 0 & 1 & -4 & \ddots \\ \vdots & & & -4 & 1 \\ 0 & & & 1 & -4 \end{matrix}
&
\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & \ddots & \\ & & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{matrix}
\\ \hline
0 & 0 &
\begin{matrix} 1 & 0 & & 1 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & \ddots & \\ & & 1 & 0 \\ & & 0 & 1 \end{matrix}
&
\begin{matrix} -4 & 1 & 0 & 0 & \\ 1 & -4 & 1 & 0 & \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ & & \ddots & 1 & 0 \\ & & 1 & -4 & 1 \\ & & 0 & 1 & -4 \end{matrix}
\end{array}\right]
$$

Each sub-block is $(L+1)\times(L+1)$ in size

There are $(J+1)\times(J+1)$ sub-blocks $\rightarrow$ A is $(J+1)(L+1) \times (J+1)(L+1)$

Use linear algebra routines for band matrices

Names of relaxation methods include

- Gauss-Seidel
- Successive Over-Relaxation