Slowly Evolving Horizons in Perturbative General Relativity

by

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Contents

Abstract

A motivation is made for a quasi-equilibrium set of black hole thermodynamic laws on the merit that they, being analogous to thermodynamics, are more physical. Black hole formalisms are reviewed and it is determined that the trapping horizon is sufficient to formulate a slowly evolving horizon regime. A summary of this formalism and the accompanying thermodynamic laws are stated. To be slowly evolving, a horizon must meet certain general conditions and those conditions from [1] are stated. A particular spacetime, to be considered slowly evolving, must satisfy these conditions and they must translate to physically meaningful restrictions. Three spacetimes are checked to see that they are slowly evolving horizons. For all three spacetimes, the conditions are met and the zeroth and first laws are then confirmed to hold.

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Chapter 1

Statement of the Problem

Equilibrium is an indispensable concept in physics; particularly so in the field of thermodynamics. This fact is made obvious through the general acceptance and use of the term "thermodynamics" to refer to *equilibrium thermodynamics* and near or *quasi-equilibrium thermodynamics*. Equilibrium as well as quasi-equilibrium thermodynamics come encompassed in the term *classical equilibrium* thermodynamics. The adjective "classical" refers to the nature of the original formulation of the theory of thermal physics which was a phenomenological one based on macroscopic measurement that did not concern itself with assuming the underlying structure of matter or providing an explanation for the process [2]. The alternate case is then referred to as *non-equilibrium thermodynamics*.[1] Classical equilibrium thermodynamics centered itself, at least initially, on understanding the workings of heat engines and then on the study of heat exchange and work done on a system. The general laws of thermodynamics, in the equilibrium regime, are simple, physical and allow easy description of

[1]Undergraduate texts often establish the first law of thermodynamics for quasi-equilibrium states without clearly specifying the distinction between it and the non-equilibrium thermodynamics case. An appropriate convention will subsequently be followed here where equilibrium and non-equilibrium thermodynamics will be distinguished from each other as described above.

the system properties with time. Thermodynamics becomes, more generally though, a study of energy and entropy especially for non-equilibrium thermodynamics as a whole.

Thermodynamics' wide ranging application to all matter, in particular to the description of its bulk properties, makes it applicable to the study of black holes. It is not surprising that the field of black hole mechanics or thermodynamics too can benefit from considerations of the equilibrium case. The equilibrium regime by definition is a physical idealization which does not allow a description of a system's evolution. To provide such a realization, thermodynamics considers systems that undergo quasi-equilibrium processes. Carrying out the application of this formalism to black holes and considering some specific examples, is the main goal of this thesis.

This introductory chapter will serve as a brief review of some basic thermodynamics which will put into context and help motivate what is required to obtain a quasi-equilibrium formulation of the zeroth and first laws of black hole mechanics. A general discussion of black hole thermodynamics, horizons, and in particular slowly evolving horizons will further motivate the objectives here within. As well, there will be an outline of what is to follow in the main sections of this thesis.

1.1 Thermodynamics

To formulate classical thermodynamics is to deal with equilibrium situations and states. An equilibrium state is one whose properties are constant in time. Variables used to describe such equilibrium states are thus referred to as state variables. Quantities like pressure, P, volume, V, and temperature, T are examples of state variables [2]. With this said, it is important to understand and keep in mind that classical thermodynamics really applies only to equilibrium situations. Thus it does not immediately make sense to talk about such variables in non-equilibrium processes [3]. In general, a non-equilibrium process is can not be described in terms of state variables. State variables are by definition quantities of equilibrium states and not non-equilibrium states. The important variable in thermodynamics is temperature and it provides a very relevant example for a discussion about state variables in non-equilibrium thermodynamics. Temperature can not really be quantified in a non-equilibrium scenario. If one were to drastically change the internal energy of a beaker of water on a lab bench, classifying it as a non-equilibrium process, a thermometer would do little justice to describing the "temperature". Not until the source of energy was cut off or decreased, when the water came close to equilibrium, would temperature be measurable.

These restrictions, inherent of classical thermodynamics, effectively exclude a description of the thermal dynamics of a system. In other words, one can not describe the evolution of a system using classical thermodynamics. This is because the aforementioned equilibrium state variables don't exist for non-equilibrium states. While thermodynamics can be generalized to non-equilibrium thermodynamics using a field theory to allow for dynamics [4], it is not the goal here to elaborate on or even use such a formulation. However, it will be relevant shortly to discuss some generalities in using such a field theory in section 1.2. The reason is that the ultimate goal is to go from the general situation that requires field theory to a regime that can be described by classical thermodynamics.

Proceeding to formulate classical thermodynamics requires use the property temperature. The association of temperature with equilibrium is fundamental and first becomes apparent in the formulation of equilibrium thermodynamics with the statement of the zeroth law.

Zeroth Law of Thermodynamics If two systems are separately in thermal equilibrium with a third system, they are in equilibrium with each other [2].

From this law, the temperature T can be defined empirically as a function of pressure and volume that essentially provides a label for each possible equilibrium state [5, 2]. Temperature serves as an indicator as to whether or not a system is in equilibrium with another. Directly from the zeroth law comes the realization that there is a function of state of a body that will have the same value for two or more bodies that are in equilibrium with one another; that function is temperature.

While this definition limits one to using temperature only in equilibrium, the reality is that physicists do measure temperature in beakers of water as the system changes with time. What is more, they in essence use equilibrium thermodynamics to describe such situations. How this is done, in spite of the restrictions to equilibrium discussed above, is of fundamental importance. If a system undergoes a *quasi-static process* it is then possible to use thermodynamics as an approximation. Such a process is one in which the system and its variables deviate only infinitesimally from equilibrium at each instant. That is, the system moves from one equilibrium state to one infinitely near by. A reversible process is always quasi-static. The converse of this is not necessarily true [2]. If in the quasi-static regime, a physicist's thermometer will be in equilibrium with the water at each instant.

Laws of thermodynamics must also govern energy conservation in and out of the system. The First Law of thermodynamics is simply an expression of energy conservation that specifically includes thermal energy. It says that the change of internal energy of the system is equal to the head or thermal energy acquired by the system combined with the work done on the system.

First Law of Thermodynamics The change in internal energy of the system, ΔU, is equivalent to the heat going into the system minus the work done by the system. Mathematically,

$$\Delta U = U_f - U_i = Q + W, \tag{1.1}$$

where Q is the heat acquired by the system and W is the work done on the system respectively (this leaves work done by the system W negative).[3]

This law is true for any change in the system but does not allow for *physical* analysis of the changing or dynamic system. In Equation (1.1), Q and W are each independent of path which means they are not state functions. As far as trying to analyze changes, these "non-state functions" Q and W can have various possible values therefore no values can be assigned to them in an attempt to describe the system. For a given change in energy in the system from one state to another, their values are ambiguous. In this sense, Q and W are not well defined. A start in trying to make this law more dynamic is writing it as an infinitesimal law,

$$\Delta U = đQ + đW, \tag{1.2}$$

where $đQ$ and $đW$ are inexact differentials[2]. The physical reason for writing these as inexact instead of exact differential is that Q and W are not quantities that the system has but instead are a change the system undergoes. If we sum these changes around the "path" of the system, the changes will not be zero. The integral around a closed path of $đQ$ or $đW$ will not be zero.

It turns out it is possible to view the changes to a thermodynamic system i.e. Q and W as well-behaved. This is only possible under certain conditions. Those situations are inherently ones in which the system can be better described. In particular, the condition for this to be true of a process is that it must be *reversible*. More

[2] A mathematical discussion of exact and inexact differentials can be found the appendices of [2].

specifically, this process is frictionless and quasi-static.[3] Then the work, appearing in Equation 1.2 becomes configuration work. Configuration work is the characteristic of reversible processes and is defined in terms of an intensive variable and an infinitesimal of an extensive variable. There are different types of configuration work but a commonly considered one is that for compression of a gas where $đW \rightarrow dW = -PdV$ which is a state function. Since dU is already a state function the remaining term of the equation - the heat term - must also be one. Thus Equation (1.2) also undergoes the change $đQ \rightarrow dQ$. Once in a reversible quasi-static regime it is possible to write the first law in differential form in terms of exact differentials as a dynamic law; [5]

$$\Delta U = dQ + dW. \tag{1.3}$$

It is important to note though that dQ is not an exact differential over finite changes.

Changing the form of dQ is all that really remains in order to fix the first law. It can be shown that this infinitesimal is actually defined in terms of an exact sate variable T and another yet undefined exact differential. This can be mathematically treated by searching for the proper integrating factor (see [6, 2] for such treatment). The physical motivation for entropy though actually comes from the consideration of the Carnot cycle (See [5] for coverage of the Carnot cycle.). Clausius studied Carnot cycles and discovered in general that,

$$\oint \frac{đQ_R}{T} = 0, \tag{1.4}$$

where R refers to a reversible process. It was Clausius who defined this exact differential with integration factor $1/T$ as entropy dS,

$$\frac{đQ_R}{T} \equiv dS. \tag{1.5}$$

[3]Often times the frictionless aspect is assumed and the terms reversible and quasi-static are interchangeable. This will be the case here. This will be consistent then with the terminology used for black hole thermodynamics.

The notion of entropy allows the heat infinitesimal to be mathematically defined in terms of state variable. Equation 1.4 illustrates where the definition for entropy comes.

Equation 1.5 can be rearranged to give a definition for the $đQ_R$. This allows the first law to finally be written as,

$$dU = TdS - PdV. \tag{1.6}$$

This version of the first law is now written in terms of state variables only. This law, as is, represents a more physical law of energy conservation because it involves state variables and it serves quasi-equilibrium processes. Moreover, there is often a misconception regarding this equation and it is important to note the nature of this mistake. Because this equation was derived by obtaining exact differentials through restricting to reversible processes, one might hastily conclude that it is also restricted in the same way. According to Pippard, Equation 1.6 can be generalized to all differential processes; not just reversible ones. Since (1.6) is a relationship between functions of state, this equation holds for all differential changes. However, for differential irreversible changes, the terms on the R.H.S. of the equation no longer represent the non-state functions of $đQ$ and $đW$ respectively. Instead, if $đQ = TdS - \epsilon$ then $đW = PdV + \epsilon$, and thus Equation 1.6.

This final fix to the first law introduces entropy which in turn leads to the second law of thermodynamics. This law actually allows for full physical description of thermodynamics. The first law alone does not exclude certain prohibited occurrences like the moving of heat from a cold body to a warm body. Observation illustrates that these are physical processes that do not occur in nature but are not prohibited by the zeroth and first laws. The second law excludes such processes. The general second law can be developed by considering a cyclic path that connects two states by an irreversible path in one direction and a reversible path.

$$dS \geq \frac{đQ_R}{T} \tag{1.7}$$

The equal sign holds for a reversible process and the inequality holds for an irreversible one. The second law may be written as

$$dS \geq dS_I, \tag{1.8}$$

where dS_I is referred to as the entropy production.

If the system is isolated the second law can be simplified to

$$dS \geq 0. \tag{1.9}$$

The statement of the second law of thermodynamics is actually most often stated in the case of an isolated (no energy exchange with surroundings) system instead of that of the general case as described in Equation 1.8.

Second Law of Thermodynamics (Isolated) An isolated system's entropy remains the same or increases; the former for a reversible process, the latter for an irreversible one. This is equivalent to equation 1.9.

The zeroth and first laws that accompany the isolated case are quite trivial but worth mentioning due to their relevance in the first formulation of black hole thermodynamics. For the isolated case the heat exchange is zero thus the temperature change is zero and the body is at equilibrium. The first law states that because the work and heat exchange are both zero, the total change in internal energy is zero as expected for an isolated system.

1.2 Black Hole Thermodynamics

As was already mentioned, thermodynamics is a very general and broad field that describes the behavior of matter. It is not extremely surprising that laws of black holes that appeared in the early 1970s were found to be similar to the laws of thermodynamics. Four of these laws were formulated by Bardeen, Carter and Hawking [7] and contained geometrical properties of a black hole defined by an event horizon; three of which will be relevant for this thesis. It was surmised at that time that these black hole quantities were analogous with the entropy and temperature of thermodynamics. What is more, it was realized that black holes are actually thermodynamical in nature. Hawking radiation, discovered shortly after Bardeen, Carter and Hawking's paper, is a mechanism that allows a black hole to have a black body spectrum with a real temperature of $\kappa/2\pi$ where κ is called the surface gravity (when physical units are not dimensionalised the temperature is $(\frac{\hbar c}{2\pi k})\kappa$) [8]. A black hole also has an entropy that is one quarter its area, $S = A/4$ (in undimensionalised units the entropy is $\frac{kc^3}{4G\hbar}$) [8, 9]. Thus, black holes are truly thermodynamic in nature and thus the review of classical thermodynamics above becomes relevant here for the study of black holes.

These results however, must not be generalized to all black holes in any spacetime. One important detail required for this formulation of black hole thermodynamics is that the spacetime be stationary[4]. Because the spacetime is stationary, the black hole is also stationary. The black hole is also isolated since in a stationary spacetime not even the mass or energy change with time nor cross the horizon of the black hole.The black holes for which these rules apply are in equilibrium which is essentially the same

[4]A stationary spacetime is one which contains a timelike Killing vector, at least outside a black hole, or equivalently one whose metric is independent of t.

starting point of the development of classical thermodynamics.It is in this context that the laws of black hole thermodynamics are stated below.

For the isolated spacetime black hole, defined by an event horizon, it is shown in [7] that the surface gravity is constant on the horizon. Being the analogous quantity to temperature, the surface gravity κ here serves as a state variable of the black hole.

Like classical thermodynamics there is a first law which reads,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J, \tag{1.10}$$

where M and J are the mass and angular momentum of the black hole. Ω is called the angular velocity of the black hole [9]. A fundamental part of establishing these laws is using the event horizon as the defining boundary of the black hole. With the connection between event horizon area and entropy evident, an area second law is expected. The area A of an event horizon is nondecreasing, $\delta A \geq 0$ [7]. These laws are referred to as the laws of black hole classical thermodynamics, mechanics or black hole thermostatics.

To reiterate, these laws are for equilibrium cases only; specifically here they are for stationary spacetimes which include the subsets stationary black holes, isolated black holes, and equilibrium black holes, in that order. They do not apply to a dynamic situation. This is essentially the same starting point as for the development of thermodynamics. The development will be similar here except there are extra elements of the formulation that must be changed.

Because these laws are for equilibrium states only (i.e. stationary spacetimes) and they do not allow for dynamic situations, there is a need to follow a procedure similar to that followed for classical thermodynamics; a process must be carried out to obtain quasi-equilibrium laws for black hole thermodynamics. The classical black hole laws are defined for a black hole that is in turn defined by an event horizon which has properties consistent with stationary spacetimes and the equilibrium laws established there within. The black hole must be defined in a more general spacetime than a stationary one. Thus, there is a need to formulate a horizon definition other than the event horizon. Choosing how to define this horizon can be inferred from how the black hole thermostatics are formulated.

1.3 Horizons

The need to define quasi-equilibrium laws of black hole thermodynamics brings about the need to define an appropriate horizon that will alleviate the restrictions of the event horizon.

As mentioned, it is because the mechanics are defined for stationary spacetimes that the laws are not dynamic. This relates to the event horizon, which defines the thermodynamics introduced above, because it is defined for a stationary horizon. Not only that, it is defined for an "ideally isolated" spacetime. As such it is not a local object. Because a black hole has been typically defined in terms of future null infinity, an observer necessarily has to go there to determine if an event horizon even exists. Once one moves out of a stationary spacetime the event horizon no longer exists. Thus the black hole thermodynamics laws should not really be expressed [10] in terms of the event horizon.

Historically, the use of boundaries of black holes has progressed with various horizons all with different properties, some of which are more appropriate for black hole thermodynamics. The event horizon was explored by Penrose in 1968 long after Schwarzschild [11] found the first black hole solution to Einstein's equation in 1916. This formulation of defining the boundary of horizon actually is preceded by the notion of a trapped surface. A paper by Penrose [12] in 1964 introduces a two-surface called a trapped surface whose existence implies the existence of a singularity. This notion was used to define other alternate horizons.

A first attempt to remedy the horizon situation was to introduce a horizon that was dynamic. An apparent horizon [13] is a horizon that is non-stationary, defined in terms of the expansion of geodesics. In general, if a horizon is evolving and expanding some light rays may not be inside the apparent horizon at a given instant but based on the infalling mass, will certainly fall inside the apparent horizon after it expands. Even these geodesics by definition lie inside the event horizon or coinside with it if there is no inalling matter.

The problem with the apparent horizon is that it also shares with the event horizon the characteristic of having non-local properties. Much work has been done recently on the latter to solve the problem. Introduction of the isolated horizon provides the definition of a quasi-local horizon. However, it defines an equilibrium (or isolated) black hole as insinuated by its name. This type of horizon is a very restrictive one and does not allow the description of an evolving black hole. Trapping horizons and dynamical horizons on the other hand are both quasi-local and dynamic. While the aforementioned horizons are all slightly different and of varying application, they are similarly defined in terms of trapped surfaces and expansions. Of specific use here will be the trapping horizon; a horizon which retains some of the notions of apparent horizons and contains some extra conditions [14]. Such a discussion will be considered in Chapter 3.

1.4 Slowly Evolving Black Holes

With these various dynamic (or non-stationary) horizon definitions one can think of trying to formulate the laws of black hole mechanics in a local and dynamic way. Once the appropriate quasi-local horizon is identified, then an obvious approach would be to define generalized versions of the parameters that appear in black hole thermostatics. One could imagine defining for example a fully dynamic generalization to surface gravity (i.e. Hayward's trapping gravity [15]) in order to produce fully dynamic laws. It is not the aim here to discuss the prospect of such an approach. Instead, it is the goal to establish a quasi-static or quasi-equilibrium regime for black hole mechanics. The motivation for a "Slowly Evolving Horizon" is to have a regime that is truly analogous to that of classical thermodynamics and thus more physical. There are myths about the first law of classical thermodynamics not being generalizable to dynamic black holes as cited by Hayward [15]. These laws can in fact be generalized to black holes since Equation 1.1 is simply an energy conservation law which should follow for black holes as well. What is true is that it is in general not possible to write an infinitesimal first law analogous to Equation 1.3 for a fully dynamic black hole. However, it is possible to write such a law for a black hole that is in quasi-equilibrium process or slowly evolving. In fact it is only the slowly evolving case which permits a truly analogous set of black hole thermodynamic laws.

Pursuing the slowly evolving formalism of black hole thermodynamics will have benefits over the static or dynamic versions of the same. There is great application here for slowly evolving horizons in numerical relativity. In particular it will become drastically simpler mathematically, and coincidentally computationally, to describe the stages after black hole merger, for example, once assuming the slowly evolving regime.

1.5 Outline

The main goal of this thesis is to present the characterization of a slowly evolving horizon as introduced by Booth and Fairhurst [1] and to give some physical examples and calculations. In doing so, it becomes very relevant to give a mathematical description of the horizons already mentioned in Section 1.4, in order to introduce the slowly evolving horizon. Both will be done in Chapter 3 along with statements of the black hole first and zeroth laws. In order to properly discuss these horizon formalisms there will be some General Relativity Tools reviewed in Chapter 2.

The original work of this thesis will be presented in Chapter 4 and there physical examples of slowly evolving horizons and the relevant calculations is the topic of Chapter 4. There will be presentation of a Vaidya black hole, where there is infalling null dust; a Tolman-Bondi spacetime, a spherical shell collapse; and finally a tidally distorted black hole. For each section the spacetime will be described and their respective metrics will be defined. Each will also contain derivation of the specific conditions for slow evolution and a validation of the zeroth and first laws. The calculations will attempt to validate the black hole thermodynamics of slowly evolving horizon

A conclusion chapter will summarize the thesis.

Chapter 2

General Relativity Tools

As discussed in §1.3, when studying black hole thermodynamics a necessary first step is choosing a definition for a black hole. A proper treatment of the different black hole horizon definitions, which follow in the next chapter, requires a discussion of some mathematics that arise in trying to define the boundary of a black hole.

The boundaries of black holes are necessarily hypersurfaces. These hypersurfaces sit in the four-dimensional spacetime for which general relativity theory is defined. When one discusses the physics of these horizons or hypersurfaces, the tensors and vectors associated with Einstein's equations, namely the metric, have to be 'brought' from the spacetime to the hypersurface. To discuss an object on the surface of the horizon, one needs to know how that tensor is defined from those in the spacetime.

Hypersurfaces in general can be thought of as generated by a particular set of curves. This set is a specific set known as congruence and will be described below. A particular theorem says; if a congruence has a vanishing rotation, then the congruence is orthogonal to a hypersurface. While the rotation of a congruence tells whether it is surface forming, other properties of such congruences, like the congruence, are telling of the gravitational field of the spacetime and in such a way are useful for defining black hole horizons.

An example of the connection between hypersurfaces and congruences is the event horizon. The event horizon is a hypersurface. Further, it is well known that such a hypersurface is null and composed of, or more precisely defined by, a congruence of null geodesics. These concepts of congruences, of null geodesics specifically, will also be useful in defining the other horizons as well. In particular, trapped surfaces are also defined by congruences of null geodesics. Trapped surfaces compose or define other types of horizons.

A detailed, well motivated discussion of the properties of a congruence of curves as well as hypersurfaces and their mappings, are the subject matter of this chapter. Specifically the behavior of geodesics congruences will be described and the behaviors associated with the congruences that define the event horizon and trapped surfaces will be identified. In general, the formulation of a mathematical definition of a black hole and its horizon will be done using the concepts associated with geodesic congruences.

While only concepts pertaining to null geodesic congruences will be of direct relevance here, it will be useful to introduce timelike geodesics and their traits. Spacelike traits are analogous to the timelike ones and as such will not require consideration. The treatment of timelike geodesics is the simpler of the two and will serve as an introduction for the null geodesics.

This approach follows that used by Poisson [16] and much of the pedagogy used here will be similar. To start, in Section 2.1, the basics of geodesic congruences are discussed. To discuss this behavior it is necessary to introduce what is meant by deviation vector. This is described in Section 2.1.3 and it is of particular importance as it is the object used to describe the evolution of the congruence.

A review of congruences of both timelike (§2.2) and null (§2.3) geodesics will constitute separate yet similar sections. Contained in each section will be a discussion on their characteristics and behavior. An important consideration in both the sections will be the scalar expansion. As will be seen, the expansion serves as an indicator of the fractional rate of change of the size (area or volume) of the surface. The shear and rotation will also determine properties of the congruences evolution.

Important statements pertaining to geodesic congruences will appear in both these sections. A general statement of their behavior will be given mathematically through Raychaudhuri's equation. Preceding that will be a statement of Frobenius' Theorem, which tells how congruences relate to the notion of a hypersurface; in particular, null congruences define null hypersurfaces.

A discussion of hypersurfaces and mapping between manifolds will take place in the last section of this chapter. As mentioned, the tensors of the general theory of relativity are defined in the four-dimensional spacetime and must be mapped to the hypersurface of interest. This is of general fundamental importance. In particular, some horizons defined in Chapter 3 are defined using a particular map, the pullback, to the hypersurface that represents the horizon. Calculations of such objects is also required in Chapter 4. Section 2.4 discusses precisely hypersurfaces and how objects are mapped to these surfaces.

Finally, §2.5 serves to revisit congruences in terms of the non geodesic curves and in regard to hypersurfaces. This section will summarize the connection between the tensors, describing the congruences and the mapping to hypersurfaces, without assuming the curves are geodesic.

2.1 Congruences

Congruences are important because they can be used to describe flow lines of fluids or histories of null geodesics. While properties of curves that are not necessarily geodesics will be summarized later, the more immediately physically relevant congruences are those of geodesics.

2.1.1 Geodesics

A geodesic is a 'straight line' in a possibly curved spacetime. More technically it is the longest (under local conditions) possible distance between two events in a spacetime. In calculating such an extremization, distances are defined using the metric. Distance is defined along an arbitrarily parameterized curve between the two events. Of course in Euclidean space the classical straight line is recovered from this definition of a geodesic. The role geodesics play in general relativity is considerable. Since they generalize the definition of a straight line, they allow a liberation of spacetime (and attempts to describe it) from the confines of special relativity's flatness;

> The world lines of freely falling bodies in a gravitational field are simply the geodesics of the (curved) spacetime metric [17].

They provide a general mathematical description of a particle's path. In order to be geodesic a curve with tangent vector u^α must everywhere satisfy ${u^\alpha}_{;\beta}u^\beta = 0$, which assumes affine parameterization.. It is not surprising then that families or congruences of geodesics are important in general relativity and that they reveal details about the nature of the gravitational field in general.

2.1.2 Congruence

A congruence in an open region of a manifold is a set of curves where only one member passes through each given point in that region. This congruence is directly associated with a vector field as the tangents to it give a vector field in the open region for which the congruence is defined and vice versa [17].

This structure provides a mechanism for measuring the behavior of a hypersurface and how it evolves. In particular the physically interesting congruences are the congruences of timelike and null geodesics.

2.1.3 Deviation Vector

A determination of the geometry of the congruence is made possible by introducing a deviation vector ξ^α which is defined to represent a displacement from one geodesic to its nearest neighbor. The setup, as per [16], is a coordinate system $x^\alpha(s,t)$ with t as an affine parameter along the curve and with s as a label for the geodesic. Thus the family of geodesics is represented by the coordinates $x^\alpha(s,t)$ where each member of the family can be considered by keeping s constant and varying t. The vector field tangent to these geodesics is $u^\alpha = \frac{\partial x^\alpha}{\partial t}$. Because this is a geodesic, u^α satisfies the geodesic quation.

If however, t is fixed and s is varied, a new family of curves can be considered. For this family, the tangent vector field defined at the first geodesic gives a notion of a deviation vector $\xi^\alpha|_{s=0}$ where the tangent vector field is $\xi^\alpha = \frac{\partial x^\alpha}{\partial s}$. This notion can be qualified by considering a vector of this vector field at say $(s = 0, t = 0)$ as one goes to $(s = 1, t = 0)$ and how much the coordinate value has changed. The vector actually tells the rate of change of the coordinate as one begins to move to the nearest geodesic while holding t constant. This notion of deviation vector is restricted to the first geodesic in order to retain its meaning. It represents an infinitesimal displacement to the next geodesic. Accordingly, from this point it is assumed that the deviation vector is defined at the first geodesic, $\xi^\alpha = \xi^\alpha|_{s=0}$.

It can be shown that the derivative of $u^\alpha \xi_\alpha$ along the affine parameter is zero and thus is constant along some initial geodesic. The geodesics can be parameterized so that $u^\alpha \xi_\alpha = 0$. Thus the vector does represent a deviation as it is perpendicular to an initial geodesic. For instance, if proper time were held constant one could see how the geometry of the congruence changes across members of the congruence.

To write an important relation for the deviation vector, one starts by recognizing that from the definition of u^α and ξ^α above, comes the relation $\frac{\partial u^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial t}$. Then by extending this to its covariant relation, allows the generalization $\pounds_u \xi^\alpha = \pounds_\xi u^\alpha = 0$. Next evaluating the Lie derivatives presents a relation that holds for the congruence;

$$u^\alpha_{;\beta}\xi^\beta = \xi^\alpha_{;\beta}u^\beta, \tag{2.1}$$

and it represents the fact that, by construction, the deviation vector is Lie transported in the direction of u^α and vice versa [17, 16].

2.2 Congruence of Timelike Geodesics

A congruence of timelike geodesics is a congruence of curves all of which are geodesic and timelike. To be timelike, the tangent vector to a curve returns a negative value when dotted with itself. All generality can still be retained by assuming that the parameterization of the geodesic is by the proper time τ, the standard affine parameter for timelike curves. Thus the tangent vector, u^α, has the form $u^\alpha u_\alpha = -1$. Also, with this parameterization, to study the evolution is to find how the deviation of the vector changes along the parameter τ.

Because the deviation vector is transverse to the congruence flow (tangent to the geodesics), it is of interest to isolate the transverse part of the metric.

2.2.1 The Transverse Metric

By transverse, it is meant that there is no component in the direction of the flow. Thus, ξ^α should be orthogonal to u^α. The break down of the metric is into its transverse part, $h_{\alpha\beta}$, and the remaining longitudinal part, $f_{\alpha\beta}$. The transverse part can be written as

$$h_{\alpha\beta} = g_{\alpha\beta} - f_{\alpha\beta}. \tag{2.2}$$

The transverse condition can now be written as

$$u^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} u^\beta. \tag{2.3}$$

Substituting Equation 2.2 into this transverse condition here with some rearranging gives the longitudinal part of the metric as $f_{\alpha\beta} = -u_\alpha u_\beta$. Using this gives a final version of Equation 2.2;

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \tag{2.4}$$

Because the metric is chosen to be transverse to the u^α, it is also purely spatial. This is made clear when in a comoving Lorentz frame;

$$g_{\alpha\beta} \overset{*}{=} \mathrm{diag}(-1,1,1,1) \Rightarrow u_\alpha \overset{*}{=} (-1,0,0,0) \Rightarrow h_{\alpha\beta} \overset{*}{=} \mathrm{diag}(0,1,1,1), \tag{2.5}$$

where $\overset{*}{=}$ means equal only necessarily in a Lorentz frame. More tensors will be examined below that will also be transverse. Then the relevance of considering the transverse metric becomes somewhat more evident.

2.2.2 Kinematics and Expansion

To describe the evolution of the deviation vector it is possible to define a fully transverse tensor, $B_{\alpha\beta}$, as the covariant derivative of the tangent to the geodesics;

$$B_{\alpha\beta} = u_{\alpha;\beta}. \tag{2.6}$$

It is transverse because the deviation vector is transverse to the congruence flow. Verifying that this is in fact transverse requires using the transverse condition of Equation 2.3 again. It describes the evolution of the deviation vector in that it reflects the extent to which it is parallel transported along the congruence i.e. along the curve u. This in turn, represents the evolution of the congruence. Using (2.1), this notion is borne out in the relation,

$$\xi^{\alpha}_{;\beta} u^{\beta} = B^{\alpha}_{\beta} \xi^{\beta}. \tag{2.7}$$

Every tensor can be written as a sum of its symmetric and antisymmetric parts. Breaking $B_{\alpha\beta}$ down into its symmetric part $B_{(\alpha\beta)}$ and its antisymmetric part $B_{[\alpha\beta]}$ produces tensors that are more easily physically interpreted. Respectively they are the *expansion tensor* θ_{ab} and the *rotation tensor* ω_{ab}. The symmetric part can be broken down further into the trace and the tensor minus the trace. To find the trace, the transverse metric can be used to find B^{α}_{α} which is actually the *expansion scalar* θ. The traceless part of $B_{(\alpha\beta)}$ is found by subtracting off the appropriate term from the diagonal elements $\sigma_{ab} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$ and is known as the *shear tensor*. The kinematic tensor now can be fully decomposed as

$$B_{ab} = \frac{1}{3}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab}. \tag{2.8}$$

These kinematic quantities can be interpreted as follows. The expansion is fundamental for determining, in relation to massive particles, the properties of the gravitational field. It can be shown (e.g. in [16]) that the expansion is the rate of change of the volume of the cross-section of the congruences:

$$\frac{1}{\delta V}\frac{d}{d\tau}\delta V, \tag{2.9}$$

where $\delta V = \sqrt{h}d^3y$ where $h = \det[h_{ab}]$ [16].

To consider the effects of the shear and rotation tensors one can start with a slice of the congruence that forms a sphere. The shear σ_{ab} for example indicates how an initial slice of the congruence evolves along u^α and stretches or contracts the circle along the major or minor axes to make the sphere into an ellipsoid. The rotation tensor ω_{ab}, if the only non-zero part of the expansion tensor, rotates the sphere.

2.2.3 Frobenius' Theorem

To get a good understanding of Frobenius' theorem one has to be aware of the definition of hypersurface. While a hypersurface defined in terms of maps and manifolds will be presented in section 2.4, a general definition is given here. In a given space, a hypersurface is a subspace of one lesser dimension. That lower dimension is acquired at a cost of losing a direction in the space. This is consistent with the best known example of a hypersurface: a plane in three-dimensional space.

Frobenius' Theorem pertains to congruences and hypersurfaces. It characterizes congruences that, throughout the spacetime, are orthogonal to a group of hypersurfaces that make up or *foliate* the spacetime. The characterization of hypersurface orthogonal congruence is a statement which follows from the definition of hypersurface.

One can always associate with a hypersurface a normal n^α and defining equation $\Phi(x^\alpha) = c$ where c is a parameter labeling the hypersurfaces of the foliation. Explicitly then, to be hypersurface orthogonal the tangent to the congruence must be proportional to the hypersurface normal and thus,

$$u_\alpha = -\mu \Phi_{,\alpha}, \tag{2.10}$$

with the proportionality being expressed by the constant μ. Since the tensor $u_{[\alpha;\beta}u_{\gamma]}$ is antisymmetric it can be written as $\frac{1}{3!}\left(u_{\alpha;\beta}u_\gamma + u_{\gamma;\alpha}u_\beta + u_{\beta;\gamma}u_\alpha - u_{\alpha;\gamma}u_\beta - u_{\beta;\alpha}u_\gamma u_{\gamma;\beta} - u_\alpha\right)$ which by the commutativity of the covariant derivatives goes to zero. Thus the theorem of Frobenius is achieved;

$$\text{hypersurface orthogonal} \Rightarrow u_{[\alpha;\beta}u_{\gamma]} = 0. \tag{2.11}$$

This statement, along with its converse, says that a congruence with tangent vector field u_α can be characterized as hypersurface orthogonal if and only if $u_{[\alpha;\beta}u_{\gamma]} = 0$. The proof of the converse of this statement is more complicated and will not be given here.

What is significant about this form of Frobenius' Theorem is that it was proved without using the fact that the congruence was timelike or geodesic. It is simply a statement of congruences in general. A more specific version of the theorem can be obtained using these two conditions. The antisymmetry of $u_{[\alpha;\beta}u_{\gamma]}$, as stated above, can be used to write:

$$\begin{aligned} 3! u_{[\alpha;\beta}u_{\gamma]} &= 2(u_{\alpha;\beta}u_\gamma + u_{\gamma;\alpha}u_\beta + u_{\beta;\gamma}u_\alpha) \\ 0 &= 2(\omega_{\alpha\beta}u_\gamma + \omega_{\gamma\alpha}u_\beta + \omega_{\beta\gamma}u_\alpha). \end{aligned}$$

The left side is zero by (2.11). Multiplying both sides by u^γ gives $\omega_{\alpha\beta} = 0$. This is the new form of Frobenius' Theorem specifically for timelike geodesic congruence;

$$\text{hypersurface orthogonal} \Rightarrow \omega_{\alpha\beta} = 0. \tag{2.12}$$

A congruence of timelike curves with tangent vector u^a is surface forming if it has a vanishing rotation.

2.2.4 Raychaudhuri's Equation

It is now possible to find an evolution equation for the scalar expansion θ. It comes from first finding such an equation for $B_{\alpha\beta}$ and then getting its trace. The evolution of the tensor is $B_{\alpha\beta;\mu}u^{\mu} = u_{\alpha;\beta\mu}u^{\mu}$, which can be written as

$$B_{\alpha\beta;\mu}u^{\mu} = -B_{\alpha\mu}B^{\mu}_{\beta} - R_{\alpha\mu\beta\nu}u^{\mu}u^{\nu}. \tag{2.13}$$

It is not hard to see that the desired quantity is the trace of the above equation;

$$\frac{d\theta}{d\tau} = B^{\alpha\beta}B_{\alpha\beta} - R_{\alpha\beta}u^{\alpha}u^{\beta}. \tag{2.14}$$

The equation named after Raychaudhuri's follows easily;

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{ab}\omega_{\alpha\beta} - R_{\alpha\beta}u^{\alpha}u^{\beta}. \tag{2.15}$$

Individually the terms in (2.15) have some restrictions (e.g. their sign) on them and considering them together presents a picture of the evolution of the timelike geodesic expansion. In the first term, θ^2 is positive since it contains the square of θ. The second and third terms, $\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ and $\omega^{ab}\omega_{\alpha\beta}$ are both positive since they are also squares and the tensors are spatial. In the general case not much can be said about the fourth term unless some assumptions are made. Those assumptions have to do with the possible energy conditions set upon a spacetime. The subject of energy conditions involves deciding what conditions will be imposed on the stress-energy tensor of spacetime. If the *strong energy condition* holds then that means that $R_{\alpha\beta}u^{\alpha}u^{\beta} \geq 0$. Including the fact that the congruence is hypersurface orthogonal ($\omega_{\alpha\beta} = 0$; term three vanishes) and the signs of the other terms, one can write

$$\frac{d\theta}{d\tau} = -\left(\frac{1}{3}\theta^2 + \sigma^{\alpha\beta}\sigma_{\alpha\beta} + R_{\alpha\beta}u^{\alpha}u^{\beta}\right) \leq 0. \tag{2.16}$$

Thus a phenomenal and far reaching result arises from examining Raychaudhuri's equation;

> The scalar expansion θ must diminish or remain unchanged with the congruences' evolution.

Physically this means that massive particles, following timelike geodesics, if initially converging, will later converge at a greater rate; if initially diverging they will diverge more slowly. Raychaudhuri's equation in this vein is known as the *focusing theorem.*

2.3 Congruence of Null Geodesics

The null geodesics pose a slightly different setup than the timelike ones. The only difference is that the tangent to the congruences is null and is labeled ℓ^α. Now the magnitude of the tangent vector is $\ell^\alpha \ell_\alpha = 0$.

The departure from the procedure used in the timelike treatment starts with that of the transverse metric. The distinction is enough to make a significant difference and accordingly it has warranted its own section. The tensors defined in this section are analogous but not the same as those in the previous section.

2.3.1 The Transverse Metric

Finding the transverse metric here is a little more involved. Because $g_{\alpha\beta} + \ell_\alpha \ell_\beta$ does not give zero when contracted with the tangent vector ℓ^α but instead gives ℓ_α, it is not the transverse metric. The remedy for this is not conceptually straightforward without going to a comoving Lorentz frame. However, seeing how this attempted transverse metric contracts with ℓ^α in comparison with the same in the timelike case provides a solution. What is obvious is that the metric can be written as the sum of the transverse and longitudinal parts;

$$h_{\alpha\beta} = g_{\alpha\beta} - f_{\alpha\beta}, \tag{2.17}$$

where $f_{\alpha\beta}$ is the yet unknown longitudinal part. Now the contraction with the transverse metric and the tangent vector (with index α) gives $f_{\alpha\beta}\ell^{\beta} = \ell_{\alpha}$. Finding the longitudinal part using this transverse condition will fully define the transverse metric. Unlike before, solving for $f_{\alpha\beta}$ cannot be done by simply contracting both sides with ℓ_{β} since the left side goes to zero. To find the transverse part requires introducing a new general null vector from the congruence that will give a non-zero constant when contracted with ℓ_{β}. Here that vector is labeled n^{α}. Now $f_{\alpha\beta}\ell^{\beta}n_{\beta} = \ell_{\alpha}n_{\beta}$. If the cross-normalization of the two null normals is $\ell^{\beta}n^{\beta} = c$ for a particular constant c then $f_{\alpha\beta} = \frac{1}{c}\ell_{\alpha}n_{\beta}$. Considering the other transverse condition i.e. the contraction with index β, also produces $f_{\alpha\beta} = \frac{1}{c}n_{\alpha}\ell_{\beta}$. As it turns out combining both these solutions gives $f_{\alpha\beta} = \frac{1}{c}(\ell_{\alpha}n_{\beta} + n_{\alpha}\ell_{\beta})$ where c is actually arbitrary; for simplicity it is chosen to be -1 here. The form of the transverse metric is then:

$$h_{\alpha\beta} = g_{\alpha\beta} + \ell_{\alpha}n_{\beta} + n_{\alpha}\ell_{\beta}. \tag{2.18}$$

After following through the math, it turns out that this metric is transverse to both null directions and so is two-dimensional. In particular (2.18) shows that $h_{\alpha\beta}$ is also orthogonal to n^{α}.

$B_{\alpha\beta}$ again represents the kinematics of the spacetime and the evolution of the congruence;

$$B_{\alpha\beta} = \ell_{\alpha;\beta}. \tag{2.19}$$

It registers to what degree the deviation vector is parallel transported along the null geodesics;

$$\xi^{\alpha}_{;\beta}\ell^{\beta} = B^{\alpha}_{\beta}\xi^{\beta}. \tag{2.20}$$

However this tensor, as is, is not useful as it was before in the timelike case. The problem lies in the fact that the tensor is not transverse; $B_{\alpha\beta}$ is not orthogonal to n^{α}. A scheme is required here to relieve this tensor of its non-transverse components.

To remove the non-transverse components, the metric h^{α}_{μ} is used to first remove the transverse components from the deviation vector and then find its covariant derivative while removing the remaining transverse component from it.

The transverse form of (2.20) gives, after some simplifications,

$$\widetilde{\left(\tilde{\xi}^{\alpha}_{;\beta}\ell^{\beta}\right)} = h^{\alpha}_{\mu}h^{\nu}_{\beta}B^{\mu}_{\nu}\tilde{\xi}^{\alpha} = \tilde{B}^{\beta}_{\alpha}\tilde{\xi}^{\beta}, \tag{2.21}$$

where the tilde indicates transverse e.g. $\tilde{\xi}^{\alpha} = h^{\alpha}_{\mu}\xi^{\mu}$. Explicitly the deviation vector considered here must be the transverse one. In this expression, the form of $\tilde{B}_{\alpha\beta}$ is

$$\widetilde{B}_{\alpha\beta} = B_{\alpha\beta} + \ell_{\alpha}n^{\mu}B_{\mu\beta} + \ell_{\beta}B_{\alpha\nu}n^{\nu} + \ell_{\alpha}\ell_{\beta}B_{\mu\nu}n^{\mu}n^{\nu}. \tag{2.22}$$

The tensor can be broken down in the same way as before,

$$\widetilde{B}_{\alpha\beta} = \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \tag{2.23}$$

except now the equation is two-dimensional. The scalar expansion θ is the trace of the transverse of $\widetilde{B}_{\alpha\beta}$; $\theta = g^{\alpha\beta}\widetilde{B}_{\alpha\beta} = g^{\alpha\beta}B_{\alpha\beta}$. Further using Equation (2.22) gives the basic form of the expansion;

$$\theta_{(\ell)} = \ell^{\alpha}_{;\alpha}. \tag{2.24}$$

These kinematic quantities can be interpreted in the same way as for the timelike congruence. The expansion is fundamental for it tells the properties of the gravitational field. In particular it describes the field in relation to particles of light. The difference here from the timelike case is that it can be shown that the expansion is the rate of change of the area (as opposed to volume) of the cross-section of the congruences:

$$\frac{1}{\delta A}\frac{d}{d\tau}\delta A, \tag{2.25}$$

where $\delta A = \sqrt{h}d^3y$ and $h = \det[h_{ab}]$.

To consider the effects of the shear and rotation tensors one can consider slice of the congruence that forms a circle. The shear σ_{ab} for example indicates how an initial slice of the congruence evolves along u^α and stretches or contracts the circle along the major or minor axes to make the circle into an ellipse. The rotation tensor ω_{ab}, if the only non-zero part of the expansion tensor, would rotate the circle.

2.3.2 Frobenius' Theorem

The general form of Frobenius Theorem as stated in (2.11) holds true here. From that, the null form is $\ell_{[\alpha;\beta}\ell_{\gamma]} = 0$ which can be easily written in terms of $B_{\alpha\beta}$; $B_{[\alpha\beta]}\ell_\gamma$ + $B_{[\gamma\alpha]}\ell_\beta + B_{[\beta\gamma]}\ell_\alpha = 0$. Multiplication by n^γ and a rearrangement of terms produces,

$$B_{[\alpha\beta]} = B_{\gamma[\alpha}\ell_{\beta]}n^\gamma + \ell_{[\alpha}B_{\beta]\gamma}n^\gamma. \tag{2.26}$$

Substitution into Equation (2.22) easily gives $\tilde{B}_{[\alpha\beta]} = 0$ which is enough for Frobenius' Theorem for the null case,

$$\text{hypersurface orthogonal} \Rightarrow \omega_{\alpha\beta} = 0. \tag{2.27}$$

This hypersurface must be null. Recognizing here that the null vector ℓ^α is orthogonal *and* tangent to the hypersurface means that the geodesics lie with in the hypersurface. For that reason they are referred to as the *null generators*.

2.3.3 Raychaudhuri's Equation

Analogously to before, Raychaudhuri's equation, assuming the null energy condition, is

$$\frac{d\theta}{d\tau} = -\left(\frac{1}{3}\theta^2 + \sigma^{\alpha\beta}\sigma_{\alpha\beta} + R_{\alpha\beta}\ell^\alpha\ell^\beta\right) \leq 0. \tag{2.28}$$

This again gives the same interpretation; the scalar expansion, θ, must diminish with the congruences' evolution. Physically this means that massless particles, following null geodesics, if initially converging will later converge at a greater rate; if initially diverging will diverge more slowly.

2.4 Hypersurfaces

Most of the material covered in §2.2 and §2.3 involved transverse tensors. What was not discussed was exactly why the move was made to the transverse scenario and what the underlying meaning was associated with that. Generally there was a map made from one manifold to another. Accordingly, this section will cover maps of manifolds.

The precise map that the tensors undergo is one that leaves them in a hypersurface which is a particular subspace of the four-dimensional spacetime. A definition for hypersurface will be finally stated precisely. Using the terminology covered in Section 2.4.1, the mapping to and from a hypersurface will be defined. This will bring a new perspective regarding the congruence and the associated tensors. Specifically, there will be a discussion about the transverse metric and how it is regarded as the induced metric of the hypersurface.

2.4.1 Mapping between Manifolds

A congruence gives rise to an associated vector field in an open subset $O \subset \mathscr{M}$ and the converse of this is also true [17].

When dealing with hypersurfaces, one must be able to map objects of interest (e.g. the metric tensor) from the higher dimensional manifold representing spacetime to the hypersurface. Doing so requires a discussion of what is often referred to as 'pullbacks' and 'pushforwards'. Below the approach of Carrol [18] is followed closely.

To define a map between two manifolds, it is necessary to maintain generality and allow that the manifolds may be of different dimension. The manifolds $\mathscr{M}$ and $\mathscr{N}$ have coordinates x^μ and y^α respectively as described in Figure 2.1. The map ϕ between the two manifolds is an association of each and every point in $\mathscr{M}$ with exactly one in $\mathscr{N}$. The set of points belonging to $\mathscr{N}$ that have a map from $\mathscr{M}$ is called the *image* of ϕ.

The simplest object to map between manifolds is a function which is defined on $\mathscr{N}$ as, $f : \mathscr{N} \to \mathbb{R}$. Because there is a map between manifolds there is a natural way to use the function f on the manifold $\mathscr{M}$. This function is known as a pullback of f and is defined as,

$$\phi_* f = (f \circ \phi). \tag{2.29}$$

The term pullback is an appropriate choice since it 'pulls back' the function from the manifold $\mathscr{N}$ to the manifold $\mathscr{M}$; it is 'backward' because the function moves against the direction of the map ϕ [18].

Before evolving this discussion to vectors, one-forms and higher order tensors, the other type of possible map, the 'pushforward', must be considered. A function cannot be pushed forward. If there were a function defined on $\mathscr{M}$, there would be no way to take points from $\mathscr{N}$ to $\mathscr{M}$ where the points in $\mathscr{M}$ would act as input into the function since the map goes from $\mathscr{M}$ to $\mathscr{N}$. There would of course be a natural way to do so if the existing map had an inverse, ϕ^{-1}. (Thus if the manifolds are diffeomorphic then it is possible to push forward or pull back all ranks of tensors.)

A natural procedure for a pushforward of vectors, however, is possible. Using the procedure for the pullback of a function, allows a definition of vector pushforward. For a vector V at a point $p \in \mathscr{M}$ its pushforward gives a vector at $\phi(p) \in \mathscr{N}$. The pushforward $\phi^* V$ is defined such that it satisfies

$$(\phi^* V)(f) = V(\phi_* f). \tag{2.30}$$

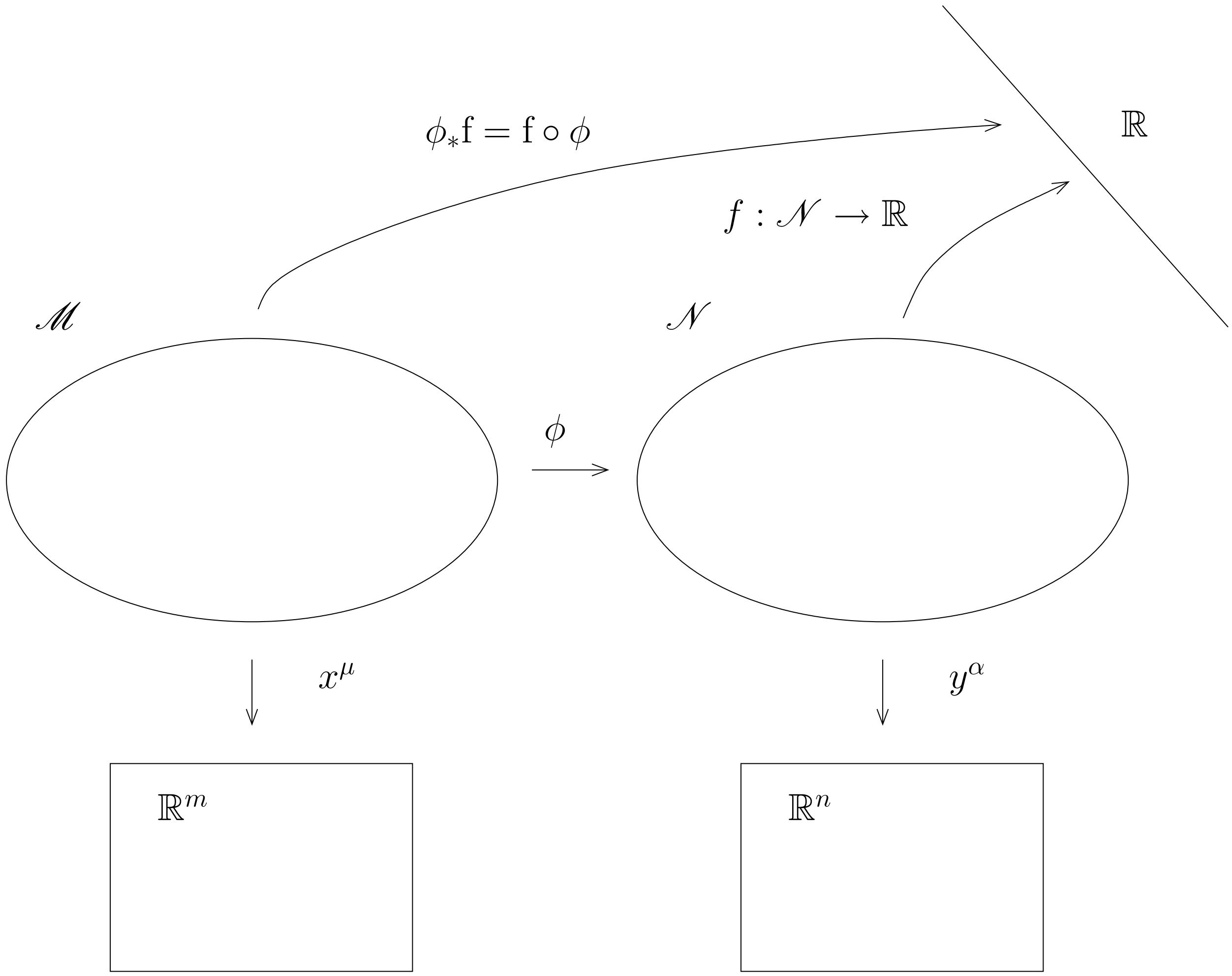

Figure 2.1: Mapping of manifolds and the objects that live on them.

In words, $\phi^* V$ is chosen such that its action on the function f (which is, again, defined on $\mathscr{N}$) gives the same result as the vector V acting on the pullback of f, $\phi_* f$. It is a pushforward in the sense that an initial vector V associated with a point $p \in \mathscr{M}$ becomes a vector $\phi^* V$ at $\phi(p)$. It is truly a vector since it satisfies the vector axioms at $\phi(p)$ [17]. This pushforward looks a lot like the transformation of a vector. For a discussion on this see [18].

Exploring this similarity of the vector pushforward to vector coordinate transformation proves very important since it involves looking at the pushforward from coordinate perspective. Because the vectors have a basis in both $\mathscr{M}$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and in $\mathscr{N}$, $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$ it is possible to find a coordinate relation between the vector and its pushforward. Equation 2.30 becomes,

$$(\phi^* V)^\alpha \partial_\alpha f = V^\mu \partial_\mu (\phi_* f), \tag{2.31}$$

after correcting the vector and its pushforward in terms of its respective bases. The pullback function can be written as a composition using (2.29),

$$(\phi^* V)^\alpha \partial_\alpha f = V^\mu \partial_\mu (f \circ g),$$

and then using the chain rule allows a simplification from the composition derivative to the derivatives of individual maps,

$$(\phi^* V)^\alpha \partial_\alpha f = V^\mu (\frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f). \tag{2.32}$$

This is a coordinate version of the pushforward of a vector.

Dual to the vector, the one-form can be mapped, in an opposite way to the vector, as a pullback. The pullback of a one-form is defined by recalling that the one-form that once acted on a vector V gives the same result as the one-form acting on the pushforward of the vector V, $\phi^* V$;

$$(\phi_* w)(V) = w(\phi^* V). \tag{2.33}$$

The coordinate version of the pullback of the one-form is found as before for the vector pushforward;

$$(\phi_* w)^\alpha \frac{\partial}{\partial x^\alpha} f = V_\mu \partial_\mu \left(\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial x^\alpha} f\right). \tag{2.34}$$

Pulling back a (0,2) tensor is of great interest since the metric tensor falls into that category. The method for defining such a map is similar to that already used for pushforward of a one-form and nearly the same as that for the pullback of a vector. Of course a difference here is that the tensor requires more than one input; two vectors. Because it is already known how to map these vectors using ϕ or more precisely ϕ^*, these pushed-forward vectors can be used to define the pulled back tensor,

$$\phi_*(T(V^{(1)}, V^{(2)})) = T(\phi^* V^{(1)}, \phi^* V^{(2)}), \tag{2.35}$$

as the operation of the tensor T on the pushed-forward vectors $\phi^* V$.

For all the higher rank tensors, contravariant and covariant, the technique for defining them follows in a similar way. However mixed rank tensors cannot, in general be pulled back or pushed-forward [18].

Induced Metric

The pullback of the two-form defined in (2.35) provides a means for moving the spacetime metric to the hypersurface. A spacetime metric in $\mathscr{M}$ pulled back to the hypersurface is known as the induced metric. In particular, the coordinate form of (2.35), as yet unstated, gives the form of the induced metric.

This is consistent with the induced metric as laid out by Poisson. It is intrinsic to the hypersurface as it can be found by imposing restrictions on the line-element which induces the new metric.

$$\begin{aligned} ds^2_\Sigma &= g_{\alpha\beta} dx^\alpha dx^\beta \qquad (2.36) \\ &= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \\ &= h_{ab} dy^a dy^b \qquad (2.37) \end{aligned}$$

A proper definition of the hypersurface induced metric in terms of maps will occur in the next section.

Another notation for a pullback, which can be used to note the *induced metric* in terms of the spacetime metric is as follows, $\phi g_{ab} = \underleftarrow{g}_{\alpha\beta}$. The same notataion can be used to describe the pullback of the covariant derivative i.e. $\underleftarrow{\nabla}_a$.

2.4.2 Hypersurface

Hypersurfaces can be defined in terms of an $(n-1)$-dimensional submanifold $\mathscr{S}$ and an embedding $\phi : \mathscr{S} \to \mathscr{M}$. A hypersurface in a manifold $\mathscr{M}$, is the image $\phi(\mathscr{S})$ of $\mathscr{S}$ [13]. A standard four-dimensional spacetime manifold will give a three-dimensional hypersurface.

In the coordinate sense a hypersurface is formed by introducing a constraint,

$$\Phi(x^\alpha) = 0, \qquad (2.38)$$

or issuing an equivalent set of parametric equations,

$$x^\alpha = x^\alpha(y^a). \qquad (2.39)$$

A coordinate definition is consistent with a mapping definition of hypersurface. Because the map ϕ has associated with the pullback of the vectors through the map ϕ_* which maps from T_p to $T_{\phi(p)}$, it is easy to see how a direction is lost. There must be some form n_a belonging to $T^*_{\phi(p)}$ such that for a vector $X^a \in T_p$ from $\mathscr{S}$, $g_{ab}n^a\phi_* X^a = 0$.

The induced metric from $\mathscr{M}$ is defined on $\mathscr{S}$ and is induced by the map ϕ. If X^a, $Y^b \in T_p$ then $\phi^* g_{ab}$ is an induced metric if $\phi^*(g_{ab}X^aY^b)|_p = g_{ab}\phi_*(X^a)(\phi Y^b)|_p$.

It is possible to find a tensor that is a result of pulling back the metric from $\mathscr{M}$ and then pushing it back into $T^*_{\phi(p)}$. To do that requires $g^{ab}n_an_b \neq 0$ where the normalization can be picked so that $g^{ab}n_an_b = \pm 1$. Under this circumstance and given the subspace $H^*_{\phi(p)}$ of $T^*_{\phi(p)}$ such that all one-forms ω_b at $\phi(p)$ give $g^{ab}n_a\omega_b = 0$, then ϕ^* will be one-to-one on H^*. Thus if the inverse $(\phi^*)^{-1}$ will be defined in such a case it earns the label $\tilde{\phi}_*$. Because ϕ_* already maps contravariant tensors, this inverse can be generalized to covariant tensors (e.g. the metric tensor) $(\tilde{\phi}_* T^{a\ldots b}_{c\ldots d} n_a = 0$ and $(\tilde{\phi}_* T^{a\ldots b}_{c\ldots d} g^{ce} n_e = 0$.

With this map one can now move the metric from the spacetime manifold $\mathscr{M}$ to the hypersurface $\phi(\mathscr{S})$. The new metric is $\tilde{\phi}_*(\phi^* g_{ab})$. Knowing that this metric must satisfy $\phi^* h_{ab} = \phi^* g_{ab}$ and $h_{ab}g^{bc}n_c = 0$, it is possible to write the tensor in terms of the normalized normal n_a as $h_{ab} = g_{ab} \mp n_an_b$. This is now recognizable as the transverse metric from Section 2.3.1.

Thus the tensor $h^a_b = g^{ac}h_{cb}$ is a projection operator. Showing this can be done easily with $h^a_b h^b_c = h^a_c$. It projects into a subspace $H = \phi_*(T_p)$ of $T_{\phi(p)}$. From T_p to $H_{\phi(p)}$ the map ϕ_* is one-to-one so that the map $(\phi_*)^{-1}$ exists in this case and is labeled $\tilde{\phi}^*$. Like before, because it is already possible to map covariant tensors from $\phi(\mathscr{S})$ to $\mathscr{S}$, this inverse function $\tilde{\phi}^*$ can be expanded to include tensors in general.

Now looking at the properties of $\tilde{\phi}^*$ and $\tilde{\phi}_*$ in composite, $\tilde{\phi}^*(\tilde{\phi}_* T^{a\ldots b}_{c\ldots d}) = T^{a\ldots b}_{c\ldots d}$ and $\tilde{\phi}_*(\tilde{\phi}^* T^{a\ldots b}_{c\ldots d}) = T^{a\ldots b}_{c\ldots d}$. This property allows one to categorize tensors on $\mathscr{S}$ and those on H on $\phi(\mathscr{S})$ as corresponding to one another. In this way, h_{ab}, the transverse metric, corresponds to the induced metric $\phi^* g_{ab}$ and can thus be referred to as the induced metric. The differences are obvious but for all intents and purposes the two are the same. The induced metric is defined on $\mathscr{S}$ and follows from the map of g_{ab} from $\mathscr{M}$. The transverse metric is actually a second map of the induced metric $\phi^* g_{ab}$ from $\mathscr{S}$ back to $\mathscr{M}$ but in the hypersurface $\phi(\mathscr{S})$.

2.5 Congruences Revisited and Summarized

As already alluded to, the reasoning for finding the transverse tensors in §2.2 and §2.3 has been overlooked. Here, a discussion of just that will be presented. An overview of the kinematics of congruences and the association with hypersurfaces will serve as summary of some of the important properties of congruences and their evolution. It is of interest now to concisely generalize the idea of expansion and other properties of congruences of curves while generalizing to those that are not necessarily geodesic since the previous treatment considered only geodesic curves.

The rationale for finding transverse tensors, as it will be seen, is that the tensors of interest are those defined on a hypersurface that must be mapped there from the spacetime. The move to a hypersurface can by understood by considering the use of the deviation vector. As already described the deviation vector describes how the congruence evolves. Particularly it measures in some sense the separation between to consecutive geodesics of the congruence. More specifically; for the timelike case the separation represented by ξ^a is between points equal distances along their respective curves from two arbitrary starting points. The properties of the deviation vector that make it inherently suited for the description of congruence evolution, also leave the tensors that describe the congruence, like the expansion; shear; nad rotation tensor, mapped to a hypersurface. The move to the hypersurface becomes evident when considering that the most relevant description of separation is between neighboring curves and not just points on the curves and realizing that this can be achieved by adding a multiple of u^a to ξ. Thus one is only concerned with ξ and points parallel to it. This means at each point q one need only deal with the space of vectors Q_q composed of the equivalence class requiring them to be different than u^a only by an added multiple. In other words one can get a projection of vectors orthogonal to u^a which can be represented as H_q of T_q composed of orthogonal vectors to u^a.

The null case is somewhat different. If one considers H_q as the subspace of T_q orthogonal to ℓ^a than in this case Q_q is not isomorphic to H_q. Instead the relevant subspace is S_q which is composed of equivalence classes of vectors which differ by a multiple of ℓ_a.

Now that it is clear that the deviation vector and the evolution tensor are mapped to a hypersurface, the kinematics can now briefly be reviewed. A comment on notation is first necessary before continuing. Instead of using the notation h_{ab}, beyond this point, the induced metric will be labelled $\tilde{q}_{ab}$ where the tilde reflects that the metric is actually the pushforward of the true induced metric $q_{ab} = \phi g_{ab}$. Again, once pushed forward this metric is transverse but $\tilde{q}_{ab}$ can in fact be viewed as the induced metric. The induced (or transverse) metric is again defined as,

$$\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a. \tag{2.40}$$

The expansion and rotation tensors can now be generally defined as the projection of the covariant derivative of the tangent to the congruence on the hypersurface defined by that congruence. Tensors on the hypersurface are defined by using the induced metric in the form of projection operator as described on page 36. The vorticity and expansion tensors are thus defined as;

$$\omega_{ab} = \tilde{q}^c_a \tilde{q}^d_b \ell_{[c;d]} \tag{2.41}$$

and

$$\theta_{ab} = \tilde{q}_a^c \tilde{q}_b^d \ell_{(c;d)}. \tag{2.42}$$

The tensors have the same physical meaning as described in §2.3. These tensors defined here are more general since they do not include the geodesic condition.

In a similar way, the scalar expansion on the surface can also generally defined as the trace found using the induced metric. The expansion of both the outgoing and ingoing are defined as

$$\theta_{(\ell)} = \tilde{q}^{ab} \nabla_a \ell_b \tag{2.43}$$

and

$$\theta_{(n)} = \tilde{q}^{ab} \nabla_a n_b. \tag{2.44}$$

The shear tensor is then simply the trace free part of the expansion tensor;

$$\sigma_{ab} = \theta_{ab} - \frac{1}{2} \tilde{q}_{ab} \theta_{(\ell)}. \tag{2.45}$$

To illustrate how all these definitions are general to curves that are not necessarily geodesic, it can be shown that the expansion definitions, as an example, correspond to the definition from sections 2.3. The outward null expansion of (2.43) expands to give;

$$\theta_{(\ell)} = g^{ab} \nabla_a \ell_b - \ell^a n^b \nabla_a \ell_b - \ell^b n^a \nabla_a \ell_b. \tag{2.46}$$

The second term goes to zero since from the distributive law $\ell^b \nabla_a \ell_b = 1/2 \left(\ell_b \ell^b \right) = 0$. If the congruences are of geodesics then, the third term goes to zero and the (2.24) is recovered.

These tensors will be used in the discussion of horizons in Chapter 3. In particular the horizon of interest, the trapping horizon, is defined using these definitions. These notions will be required in the main calculations of the thesis in Chapter 4.

Chapter 3

Horizons and the Slowly Evolving Case

An integral property of special and general relativity is that cause precedes effect. An event in spacetime can only have an influence on other events that can be reached via a beam of light or a null geodesic. A discussion of this is one of causal structure and inherently involves the boundaries of groups of events that have a particular causal relationship with other events; these are horizons in general relativity (e.g. Cauchy horizon). Such a discussion becomes necessary when exploring conditions for the existence of black hole horizons [1] in a spacetime and when characterizing them. This topic dominates the better part of this chapter. Among these horizon characterizations is that of the slowly evolving horizon, which is the horizon of interest in this thesis. Stating the properties of this horizon and expressing the associated thermodynamic laws is the main goal of this chapter.

To properly present these horizon definitions and the slowly evolving horizon in

[1] This is often directly associated with the existence of singularities; the event horizon case being an exception. The association with black hole horizons to singularities as well as the singularity theorem will be briefly discussed below.

particular, it must be done in the context of causal structure. Accordingly, a review of some fundamental causal structure will be presented. Immediately following, will be a discussion of the standard black hole definition, the event horizon. Doing so involves imposing some tedious mathematical restrictions on the spacetime; which will be summarized only. As well, some shortcomings of the event horizon will be discussed.

A locally defined alternative to the event horizon will be presented as the apparent horizon. Other newer characterizations of horizons will be reviewed; these are isolated, dynamical and trapping horizons. These definitions are explored with reference to their degree of locality or non-locality as well as their application to black hole thermodynamics. Considering the practicality of all these formalisms as a venue for stating the laws of black hole mechanics is paramount in making relevant the main discussion of this thesis - the slowly evolving horizon. Some concise comments to this effect will be made for each type of horizon.

As discussed in the introductory chapter, the slowly evolving classification proves very fundamental for exploring black holes in a very physical (quasi-local and quasi-equilibrium) way. The important part of this chapter will be the discussion of the slowly evolving horizon formalism. Finally, following the motivation provided in Chapter 1, there will be a statement of the black hole laws of thermodynamics introduced in Section 3.7. The procedure followed in establishing these laws will not be discussed in great detail here and though it is in many ways analogous to that of Section 1.1 there are important differences that warrant mention.

3.1 Event Horizon and Causal Structure

Black holes can be regarded as a trademark of general relativity. This is true because they have a special property. They are causally isolated from the surrounding universe. Events inside a stationary black hole can be connected to those outside in only one way, that is events outside can effect those on the inside but not vice versa. The boundary of that region is the event horizon. A mathematical formulation of this definition is made using the well developed theory of causal structure in general relativity.

3.1.1 Causal Structure

In order to classify sets of events into causal relationships relative to another event or set of events, definitions of some standard terms are given below. Following Wald [17], the *chronological future* of an event $p \in \mathscr{M}$ is defined as $I^+(p)$; the set of all events which, starting from some event p, can be reached by a future directed timelike curve. An obvious and similar term is defined for the chronological past, $I^-(p)$. This definition includes only massive particles (timelike curves) and does not allow for photons. Events can also be connected by photons. The *causal future* of an event, $J^+(p)$, is the set of events that, from p, can be reached via a timelike or null future-directed curve. Again, a similar definition for the past case exists as $J^-(p)$. Both pairs of sets, causal and chronological, can be generalized to include a definition for not only a single event p, but a set of events S. For example; $J^+(S)$ is the union of causal futures of all the points $p \in S$.

A set $S \subset \mathscr{M}$ is achronal if there are no $p, q \in S$ such that $q \in I^+(p)$ or equivalently $I^+(S) \cap S = \emptyset$, with $\emptyset$ of course denoting the empty set. Thus, no two points in $\mathscr{M}$ are timelike separated. Any two points belonging to S can only be connected via a

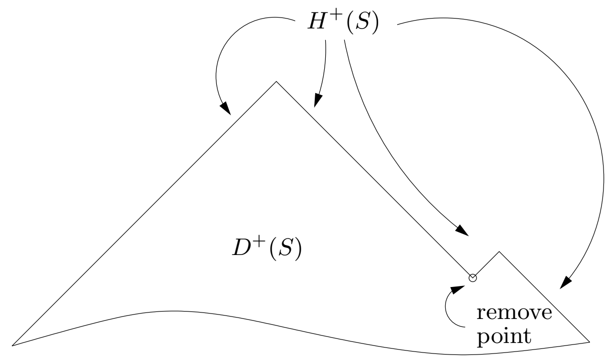

Figure 3.1: This illustrates the domain of dependence, $D^+(S)$ for the case of a removed point. The Cauchy horizon is $H^+(S)$.

null or spacelike curve.

To define a crucial concept, the Cauchy surface, a definition for the domain of dependence, is required. The domain of dependence of a set of events S (S must be closed and achronal) is the union of the past and future domains of dependence.

$$D(S) = D^+(S) \cup D^-(S), \tag{3.1}$$

where the future *domain of dependence* of S is

$$D^+(S) = [p \in \mathscr{M} \text{such that all past inextendible causal curves through p intersects } S] \tag{3.2}$$

and the future case is analogous. These domains are also referred to as the past and future Cauchy development of S.

A *Cauchy surface* is a closed achronal set Σ, the domain of dependence of which is equal to the the manifold $\mathscr{M}$;

$$D(\Sigma) = \mathscr{M}. \tag{3.3}$$

The Cauchy surface, for all intents and purposes, can be viewed as an instant in time. Thus a Cauchy surface is an important object in general relativity. Spacetimes which contain such a surface are classified as *globally hyperbolic.*[2] Σ can be seen as an instant in two ways. (i) It is achronal; thus Σ is a null-or-spacelike structure in the sense that no particle moving less than the speed of light can coincide with any two events on the surface. (ii) Its entire past and future are predictable.

Black Hole: The Event Horizon

A [stationary] black hole is defined by distinguishing between those regions of spacetime from which null geodesics can reach future null infinity and those from which they cannot. The rigorous definition of a black hole requires some constraints on the spacetime. Attempting to mathematically identify the region of a black hole in a general spacetime illustrates the problem and provides a motivation for requiring these conditions.

If the black hole is defined in a general spacetime $(\mathscr{M}, g_{ab})$ it will be a subset of the spacetime manifold, $A \subset \mathscr{M}$. Particularly, all the points p in the black hole region A will have their causal future inside the black hole. Since the union of subsets is still a subset, that is equivalent to $J^+(S) \subset A$. This subset fits the definition of a black hole as an object where light particles do not escape. While this definition seems reasonable at first glance, upon reflection one can see that the causal future of any set in a general spacetime is a black hole according to this definition. One could always give the causal future of any given set the label A; thus a black hole. In other words, any spacetime will contain a black hole; Minkowski spacetime included.

This definition does not confine the black hole to a particular region of spacetime - as can be seen in the Minkowski spacetime. The Minkowski spacetime, using the above definition, would produce a black hole that contains the entire spacetime. A

[2] The definition of globally hyperbolic used here is different from that used in Hawking and Ellis [13] and instead follows that of Wald [17]

proper definition would logically come from restricting the set to define a black hole from reaching a particular region of spacetime in a physically meaningful way. If one considers a Schwarzschild black hole, the part of spacetime that makes up the black hole is not the entire spacetime but is actually confined to a region of spacetime where r is small. It will be shown shortly that with the correct physical restriction and 'transformation', a given spacetime can be represented by one in which it is clear whether the original spacetime contains a black hole - in particular that the Minkowski spacetime does not contain a black hole but the Schwarzschild does.

A well posed mathematical definition for a black hole with an event horizon exists in a spacetime that is *asymptotically flat*. [3] The flatness refers to the zero curvature associated with an empty spacetime; it falls off asymptotically away from the region of interest. Asymptotically flat spacetimes mimic Minkowski spacetime far away from an object of interest. These spacetimes are used to represent isolated systems in general relativity. Isolated systems in physics involve physical constraints and thus asymptotic flatness is associated with a physical restriction. Generally, being in such a spacetime as this, means that the gravitational field goes to zero as one gets further away from the system. Limiting the gravitation field is a physical restriction. It is in this way that, when asymptotically flat, a spacetime provides a notion that allows distinguishment of a region for which no escape can occur. Having a rigorous definition of what escape is, is ultimately what the mathematical definition provides for a black hole.[4]

The notion of infinity in particular, *future null infinity*, is a trait of asymptotically

[3]The definition of asymptotically flat used here is the same used in [17] and it is important to note that it is slightly different than the one used in [13]

[4]Another necessary constraint is that the spacetime must be strongly asymptotically predictable which includes being asymptotically flat as well as posses a Cauchy surface in the unphysical spacetime or is *globally hyperbolic*. (See [19] or [17] for a formulation of the Cauchy condition for being strongly asymptotically flat.

flat spacetimes and is used to define a black hole as a region for which nothing can escape. It is useful to first look at Minkowski spacetime and how it is transformed to asymptotically flat as a simple example. As it turns out, in order to use this notion of infinity it is necessary to extend the Minkowski spacetime into a larger (unphysical) spacetime. Doing this requires a conformal transformation. While the procedure of moving to an unphysical spacetime through a conformal transformation is technically important, it will not be discussed here.[5] In short though, it involves adding in points at "infinity". A figure best represents the notion of conformal infinity that results from the appropriate conformal transformation. Figure 3.2 illustrates that the boundaries of the region O are the infinities. Infinity can be divided into different regions, namely past timelike infinity, i^-, past null infinity , $\mathscr{I}^-$, spatial infinity, i^0, future null infinity, $\mathscr{I}^+$; and future timelike infinity , i^+.

For the Schwarzschild spacetime, it is possible to undergo an analogous conformal transformation and produce an asymptotically flat spacetime with a future null infinity. For this transformation, it is useful to illustrate how to obtain the conformal or Penrose-Carter diagram from a series of coordinate changes while using the Schwarzschild metric. This is carried out in [16]. The resulting diagram is depicted in Figure 3.3.

Comparing the two figures of the Minkowski and Schwarzschild conformal diagrams presents the way in which future null infinity can be used to define a black hole. The diagrams can be used to easily see what region of the diagram cannot reach future null infinity. The causal past of future null infinity, $J^-(\mathscr{I}^+)$, should be, by definition, a region of the spacetime where events can reach future null infinity by a null geodesic. Thus anything not in this region must be inside a black hole. Figure 3.2

[5] See Appendix D of [17] for a good discussion of conformal transformations and Chapter 11 of the same for one on exactly how a conformal transformation gives the Minkowski conformal infinity.

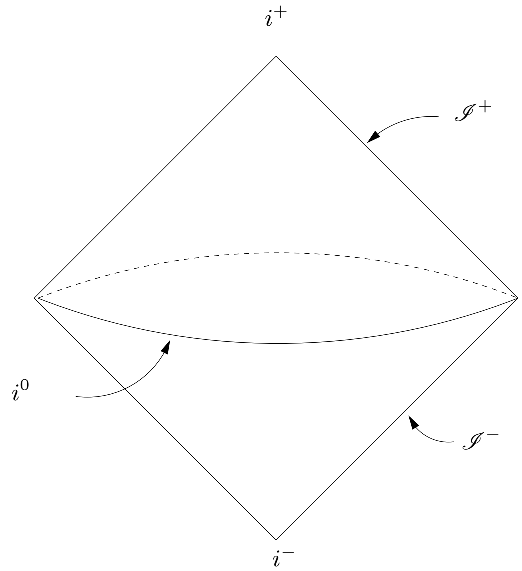

Figure 3.2: This represents a conformal transformation of the Minkowski spacetime. The boundaries of the spacetime are labelled as described in the main text.

shows that this set is the entire spacetime for a Minkowski spacetime and no region of the spacetime lies outside this set. However this region of the Schwarzschild spacetime, as in Figure 3.3, does not compose the entire spacetime. Thus, after making the conformal transformation, it is clear that the Minkowski spacetime does not have a black hole whereas the Schwarzschild spacetime does. This is as expected. The region containing the black hole is defined generally - for any spacetime - by using the future null infinity of that spacetime's conformal diagram. It has now been shown that the future null infinity, $\mathscr{I}^-$, and asymptotically flat spacetimes allow for a clear definition of a black hole.

With all this formulation out of the way, it is now possible to define a general

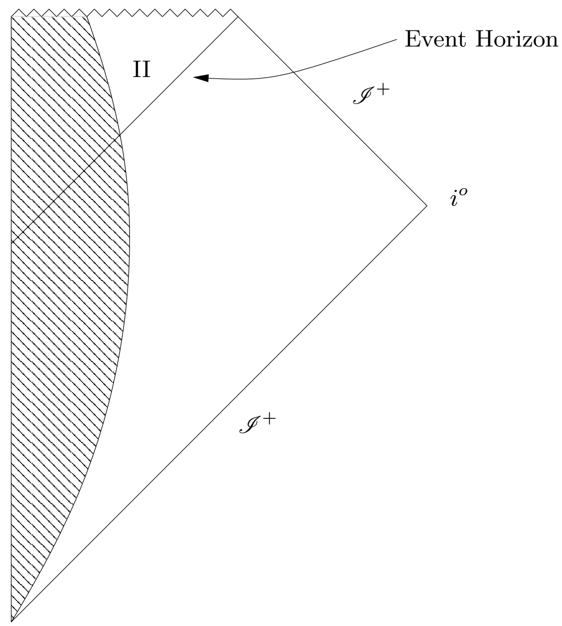

Figure 3.3: This represents a conformal transformation of the Schwarzschild spacetime. The shaded region represents the matter of the spacetime. The region labelled II is inside the event horizon and is outside $J^-(\mathscr{I}^+)$.

black hole in a strongly asymptotically predictable [6] spacetime.

$$B = \mathscr{M} - J^{-}(\mathscr{I}^{+}). \tag{3.4}$$

Equation 3.4 says that the black hole region includes all points in the manifold minus those that makeup the causal past of the future null infinity. This is equivalent to a physical notion of a black hole; that is, an event inside the black hole can never be causally connected to events at future null infinity. The boundary of B is called the event horizon.

$$H = J^{-}(\mathscr{I}^{+}) \cap M \tag{3.5}$$

With this mathematical notion of a black hole it is possible to discuss the geometry of the event horizon. It turns out that the event is a null hypersurface. [20]

This definition is very non-local in that it refers to future null infinity, $\mathscr{I}^{+}$. One can talk about whether a black hole is present at a given time, Σ, by considering $B \cap \Sigma$. This requires a notion of the entire space at that instant which is certainly non-local spatially. Furthermore to find the entire set B requires a notion of the entire spacetime. One would need the spacetime metric for all time. Discussions of event horizons, for this reason, often describe them as teleological objects.

A black hole defined with an event horizon is satisfactory for mathematical general relativity but because of its global nature it is impractical for real observers. For example, if an observer were to certainly be 'captured' at some time by a black hole resulting from a future gravitational collapse, then that observer would be sitting inside the event horizon and thus considered to be already 'in' the black hole. In this way, the event horizon is not a physical definition. What is more, because observers are not omniscient, it is not possible to even determine the existence of the event

[6] Again, this definition follows [17] which as a different definition than that used by [13] which stems from their differences in the term asymptotically flat

horizon for the reasons just described above.

The global nature of the event horizon and these physical problems with the event horizon are motivations for an alternate black hole horizon definitions.

3.2 Apparent Horizon

While the most popular definition of a black hole involves the event horizon, there are other horizons that may be used. The apparent horizon is actually a different definition of a black hole, as are the other horizons which will be described below. Introduced in 1973 [13], the apparent horizon was the first alternate, more physical horizon definition to that of the event horizon. The term black hole generally used to describe an object from which no light (matter) can escape due to its sizable gravitational field. The event horizon does not really distinguish why matter does not escape to infinity but instead simply identifies a region of space for which they do not. One of the first mathematical characterizations of a black hole was done using what is called a trapped surface. The definition of trapped surface that follows in Section 3.2.1 allows for a more physical definition of an object for which light will not escape. Roger Penrose first introduced the trapped surface in reference to gravitational collapse [20].

As it turns out the trapped surface is an indicator of a singularity. While a discussion of a singularity is strictly limited here and only qualitative, it is worth noting that the singularity can be tied closely to a black hole; thus a relevant consideration for the definition of a black hole and its associated horizon. In somewhat loose terms, a singularity is a region in spacetime where the curvature is unbounded. There are actually various singularity theorems [17] which all have various components; more than one considering trapped surface. However, for the purpose of this thesis, it will simply be said that the trapped surface acts to indicate the existence of a singularity. By tying the trapped surface to a black hole, there comes an inherent connection between a black hole, gravitational collapse, and a singularity. This is something the event horizon does not possess.

The trapped surface concept was first applied for construction of a horizon definition [21, 13] to define the apparent horizon. While many of the horizon definitions explored here use the notion of trapped surface in their definition, the apparent horizon was the first.

3.2.1 Trapped Surface

A *trapped surface* is a (smooth, closed, connected) spacelike two-surface in spacetime that contains on and inside it, a region from which no matter can escape. The characteristic used to define the trapped surfaces is the expansion of null geodesics. Normally in an empty spacetime or one with an insignificantly massive object, light emitted radially outward from a sphere $\mathscr{T}$ will form two spheres after an elapsed time, one from the ingoing rays $\mathscr{T}_1$ and one from the outgoing rays $\mathscr{T}_2$, where the areas of $\mathscr{T}_1$ is less than the area of $\mathscr{T}_2$. The typical aspect and more commonplace scenario here is that the $A_{\mathscr{T}_1} < A_{\mathscr{T}_2}$. However, in the presence of an object with sufficient mass, both of the areas of $\mathscr{T}_1$ and $\mathscr{T}_2$ will be less than $\mathscr{T}$. In this case $\mathscr{T}$ is a trapped surfaces and any particles lying on it will be subsequently trapped. Using the fact that the expansion $\theta_{(n)}$ of the ingoing null geodesics with tangents n_a, and the expansion $\theta_{(\ell)}$ of the outgoing null geodesics with tangent ℓ_a, represent the fractional rate of change of the area of the null hypersurfaces, it is possible to mathematically define the trapped surfaces. A closed (smooth, connected) two-surface is a *trapped surface* if $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$. A slight generalization of the trapped surface is the *marginally trapped surface* where the closed two-surface satisfies $\theta_{(\ell)} \leq 0$ and $\theta_{(n)} \leq 0$ [13].

In strongly asymptotically predictable spacetimes, all trapped surfaces are present inside the black hole. This property of trapped surfaces helps build the criteria for a black hole definition that does not require the future development of a spacetime.

3.2.2 The Horizon

For the apparent horizon, the spacetime constraints are not the same as before as there is not a requirement to have to be strongly asymptotically predictable. However, it will be beneficial to consider the apparent horizon in such a spacetime. [7] To first define the horizon requires covering some mathematical technicalities associated with trapped surfaces.

If there is a subset C of Σ that is closed and forms a three dimensional manifold with two dimensional boundary $S = \dot{C}$ and has $\theta_{(\ell)} \leq 0$ then S is an *outer marginally trapped surface* and C is a *trapped region* on the Cauchy surface Σ [17]. The *total trapped region* of a Cauchy surface is the closure of the union of all trapped regions on the Cauchy surface. The *apparent horizon* is then defined as the boundary of the total trapped region $\mathscr{T}$,

$$\mathscr{A} = \dot{\mathscr{T}}. \tag{3.6}$$

An important result is that the apparent horizon is an outer marginally trapped surface with vanishing expansion $\theta_{(\ell)} = 0$.

An event horizon always lies outside or on the apparent horizon [17, 10]. The apparent and event horizon coincide for the case of a stationary spacetime.

[7]Some sources to not make it clear that the spacetime need not be asymptotically flat. As well, some sources suggest that the Cauchy surface on which the apparent horizon is defined need be asymptotically flat. It does not appear that this is necessarily true.

The apparent horizon is much more local and physical than the event horizon. It is actually “local in time”. There is no need to refer to future null infinity. However, there is a need to know the entire instant that is being considered. For that reason there are still locality problems with this type of horizon. The requirement for a choice of the Cauchy surface or instants in time of the spacetime not only the makes the apparent horizon non-local but also causes other problems. In certain spacetimes (e.g. the Schwarzschild spacetime) the slicing can be chosen in such a way that no apparent horizon is present. This type of horizon requires placing restraints on the entire spacetime by requiring this choice of slicing.

3.3 Isolated Horizon

An isolated horizon is defined with an intention to retain properties of event or apparent horizons that characterize a black hole but without reference to asymptotic flatness or the whole space-time. A main motivation for this is to make the definition of the horizon truly local while maintaining the ability to define mass and angular momentum thus allowing the formulation of the thermodynamical laws. In particular, the physical situation that the isolated horizon is intended to represent is a black hole in equilibrium (i.e. non-expanding) with a possibly non-stationary surrounding spacetime.

In view of that, and knowing that the event horizon is a null hypersurface, the most physically relevant structure of the isolated horizon is a null 3 dimensional submanifold. Actually when considering the world tube of apparent horizons, it is possible to show that the apparent horizon is in general, spacelike and when in equilibrium, it is null. It follows that, the isolated horizon is closely related to the apparent horizon except it describes the world tube of horizons is null and thus in equilibrium [22]. Reasons exactly why it is null can be taken from the reasoning in [23]. That is the first of four mathematical conditions in the definition of an *isolated horizon* [24, 19]:

1. H is a null three-surface
2. $\theta_\ell = 0$ for null normal ℓ^a
3. $-T^a_b l^b$ is future directed causal
4. if $\omega_a = -n_b \underleftarrow{\nabla}_a \ell^b$ then $\pounds_\ell \omega_a = 0$

Condition two means that all the cross-sections in H are marginally trapped. Condition three is simply just an energy condition on the horizon. Actually it is just the dominant energy condition with, instead of the future directed timelike vector, a future directed null normal. This basically ensures that a "null observer" will not see negative energy. These first three conditions qualify what is technically called a non-expanding horizon. The fourth condition technically makes for a weakly isolated horizon but will be referred to simply as an isolated horizon here.

While this horizon is quasi-local it does not allow one to generate the thermodynamic laws as outlined in Chapter 1. The isolated horizon is restricted to describing equilibrium situations; isolated ones in particular. Before formulating the quasi-local, quasi-equilibrium laws of black hole thermodynamics one must formulate a horizon that is dynamic and not restricted to equilibrium.

3.4 Trapping Horizon

The trapping horizon keeps the characterization that a black hole is defined using the notion of a trapped surface. The main difference here is that the three-surfaces that

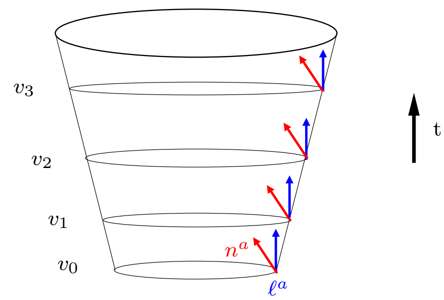

Figure 3.4: With a foliation on H of two-spheres H_v, there are null normals ℓ_a and n_a

are the horizon are not (necessarily) null surfaces. It is this loosening of the horizon definition that allows the black hole to be in a non-equilibrium state. It is possible to argue intuitively that the equilibrium state should be null and that the evolving non-equilibrium state leaves a horizon that should be spacelike, since the null surfaces, as in the case of the event horizon, isolated horizon, and (obviously) equilibrium apparent horizon, are all equilibrium states. With the definition of trapping horizon, it is possible to rigorously show just that.

Here, the type of trapping horizon of interest is a future outer one. A *future outer trapping horizon* (FOTH) is 3D submanifold H of spacetime foliated by spacelike 2-spheres H_v (v is the foliation parameter) with future directed null normals $n \cdot \ell = -1$ (standard cross normalization), as in Figure 3.4.They have the following properties:

FOTH1. $\theta_{(\ell)} = 0$,

FOTH2. $\theta_{(n)} < 0$,

FOTH3. $\pounds_n \theta_{(\ell)} < 0$.

Condition 2 here ensures that the horizon is a future horizon, as opposed to the past. This is analogous to the difference between the event horizon being the boundary of a black hole (future event horizon) and that of a white hole (past event horizon) and the rest of spacetime (see earlier Section 3.1.1). Condition 3 gives the horizon its 'outer-ness'. Characteristically this means that the points just inside are trapped while those outside are not. This captures the idea that the horizon contains trapped surfaces just inside it.

So now the definition used for the horizon is a dynamic one which includes both equilibrium (null) and non-equilibrium (spacelike) states. This is one extremely important property of the horizon. The horizon definition is local both spatially and temporally. In comparison, the apparent is spatially local and one still needs to know about the entire instant for a given instant in time.

3.5 Dynamical Horizon

While a slowly evolving horizon is defined in terms of the trapping horizon, it is relevant to define the dynamical horizon. Contrasting this horizon with the trapping horizon, the dynamical horizon does not impose conditions on the evolution of fields in the directions transverse to H. Also, the horizon must be spacelike [24].

A *dynamical horizon* (DH) is a 3D submanifold H of spacetime foliated by spacelike 2-spheres H_v where;

1. H is spacelike,
2. $\theta_{(\ell)} = 0$,
3. $\theta_{(n)} < 0$.

The relationship between the trapping horizons can be stated explicitly. A spacelike FOTH is a DH for which $\pounds_n \theta_{(\ell)} < 0$. While this horizon is also quasi-local, it does not include the equilibrium case since it is spacelike. This leaves the trapping horizon as the horizon to formulate a slowly evolving horizon.

3.6 Slowly Evolving

Now it is possible to exploit the fact that the trapping horizon covers both equilibrium and non-equilibrium states and is quasi-local. Before beginning to motivate or state the definition/conditions for a slowly evolving horizon however, it is important to reiterate the plan. The primary motivation for establishing a slowly evolving horizon comes from Chapter 1 and in particular comes from Section 1.1 regarding classical thermodynamics. For black hole thermodynamics, we also want to have state variables that describe the system. In Section 1.1, the regime for a quasi-equilibrium thermodynamics was simply stated to be reversible or quasi-static. Though this qualifies as being very slow, it does not quantify exactly what is meant by slowness. For black hole thermodynamics the aim is similar except the regime of quasi-static or slowness is to be quantified geometrically.

Once the regime is established and the conditions for slow evolution are explicit, the laws are then obtained using Einstein's equations. This is different than the laws of classical thermodynamics. There the first law comes from general conservation laws of energy and the zeroth law follows from the definition of temperature and equilibrium. In Section 3.7 the laws will be stated that arise form Einstein's equations and using the slowly evolving parameters of the horizon.

There is a way to characterize exactly to what degree a black hole is away from equilibrium using the geometry of the trapping horizon. There exists, for a given scaling of the null normals ℓ_a and n_a, an expansion parameter C on H such that $\mathcal{V}^a = \ell^a - Cn^a$ is tangent to the horizon; where $\mathcal{V} \cdot \mathcal{V} = 2C$. It is intuitive from the

Figure 3.5 that the $\mathcal{V}$ represents the tangent along the horizon. The term expansion parameter is fitting since it gives an indication to whether or not the horizon is non-equilibrium and expanding or in equilibrium and not expanding. If the one assumes the null energy constraint, which will be the case here, then $C \geq 0$. As first illustrated by [15], this inequality can be established by taking Lie derivative along the horizon of the outward null expansion. By its definition this vector $\mathcal{V}^a$, should leave the outward null expansion Lie dragged along it; $\pounds_{\mathcal{V}}\theta_{(\ell)} = 0$. Using the basic properties of Lie derivative gives

$$C = \frac{\pounds_{\ell}\theta_{(\ell)}}{\pounds_{n}\theta_{(\ell)}}. \tag{3.7}$$

Now this inequality provides two distinct possibilities: equality to zero being associated with a null horizon and greater than zero being associated with a spacelike horizon. This follows directly from the fact that the dot product of $\mathcal{V}^a$ with itself is $2C$, thus the sign of C tells the sign of the dot product and hence whether the vector is timelike, spacelike, or null. Because $\mathcal{V}$ represents the vector tangent to the horizon, the sign of C tells whether the horizon is timelike, spacelike or null. What is more, Hayward [15] has shown that the FOTH is only null if and only if the shear and energy density go to zero and thus it can be said to be in equilibrium. Otherwise it is spacelike. These properties of C are very convenient and knowing them will be important later. Summarizing, in general for a FOTH $C \geq 0$ where $C = 0$ means a horizon is null *and* in equilibrium and $C > 0$ means a horizon is spacelike *and* in non-equilibrium.

It is necessary to restrict the rescaling freedom of null vectors; $\pounds_{\mathcal{V}}v = 1$ where v is the foliation parameter. This ensures that the vector $\mathcal{V}^a$ properly represents the evolution of the horizon. While it already by definition points in the right direction, it must also have the appropriate length from one cross-section to the next. Making

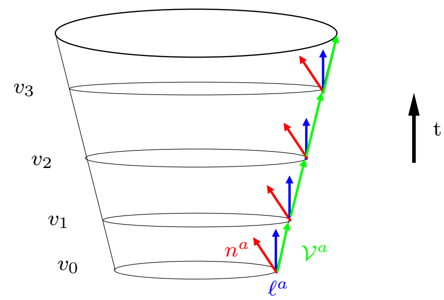

Figure 3.5: The linear combination of ℓ^a and n^a gives a vector along the horizon $\mathcal{V}^a$ for a particular scalar field C.

this choice ensures that $\mathcal{V}^a$ evolves the H_v into one another.

Now the important task is to find out what quantity can be used to determine whether the horizon is slowly evolving. More explicitly, the relevant question is, what geometric quantity/characteristic of the horizon will determine whether or not it is quasi-equilibrium? This question has already been answered by Booth and Fairhurst [1]. While the explicit details of the results will appear in a technical paper [25], the main summary and thermodynamic laws are communicated in a letter [1] and that discussion will follow here. Instead of rigorously deriving the characterization used for slowly evolving horizon, a sequence of quantities will be considered from intuitive reasoning and evaluated based on their merit.

To find slowly evolving parameters one must first find the pulled back metric on the foliated cross-sections. As was illustrated in the Chapter 2, the transverse metric can be viewed as the induced metric. With this metric one can find the area element $\sqrt{\tilde{q}}$ on the surface [16] and show that it changes as,

$$\pounds_{\mathcal{V}} \sqrt{\tilde{q}} = -C\theta_{(n)} \sqrt{\tilde{q}}. \tag{3.8}$$

To characterize slow expansion one has to look for something that indicates when the relative rate of expansion of the area A of the surface/horizon is small. Thus, one would expect $\frac{\partial A}{\partial v}/A$ to be small. This is exactly what would be intuitive for spherical symmetry but also generalizes to non-spherical symmetry. Upon integration of the above equation (3.8) for spherical symmetry over a two-surface, gives $\frac{\partial A}{\partial v} = -C\theta_{(n)}A$. Then the value for the $\frac{\partial A}{\partial v}/A$ would be $-C\theta_{(n)}$. The main problem with this candidate parameter is that it rescales with the null vectors ($\ell^a \to \alpha\ell^a$ gives $-C\theta_{(n)} \to -\alpha C\theta_{(n)}$). This aspect is unacceptable because if the characterization changes based on rescaling, one can arbitrarily change the rescaling for a given black hole to change a horizon's classification of slowly evolving.

One could attempt to avoid this rescaling problem by using the unit tangent vector to the horizon $\mathcal{V} \to \hat{\mathcal{V}} = \frac{\mathcal{V}}{\sqrt{2C}}$ where $\hat{\mathcal{V}} = \mathcal{V}/||\mathcal{V}||$. Resulting from such a move is the expression $\pounds_{\hat{\mathcal{V}}}\sqrt{\tilde{q}} \approx$ small. This gives the condition that $\left(\frac{\dot{A}}{A}\right)^2 = \frac{1}{2}C\theta_{(n)}^2 \approx \text{small}^2$ The problem here is that the normalized vector does not exist for the null case since $\hat{\mathcal{V}} = \frac{\mathcal{V}^a}{\sqrt{2C}}$ produces division by zero meaning the horizon tangent would be infinitely long. That said, $C\theta_{(n)}^2$ is still defined and does not depend on the scaling of the null vectors. Thus the condition that says the evolution of the horizon is small with respect to the scale of the horizon becomes $C\theta_{(n)}^2 \leq \frac{\epsilon^2}{R_H^2}$ where R_H is the area radius of the horizon.

Thus the main properties can now be stated for a slowly evolving horizon. To be a *slowly evolving horizon* (SEH), a horizon must satisfy the following conditions;

SE1. $C\theta_{(n)}^2 \leq \frac{\epsilon^2}{R_H^2}$,

SE2. $\sqrt{2C} \approx \epsilon$,

SE3. $|\pounds_{\mathcal{V}}\omega_a| \leq \frac{\epsilon}{R_H^2}$ and $|\pounds_{\mathcal{V}}\theta_n| \leq \frac{\epsilon}{R_H^2}$,

SE4. $|\tilde{\mathcal{R}}|$, $|\widetilde{\omega}|^2$, $|\sigma^{(n)}|^2$ and $T_{ab}n^a n^b \approx \frac{1}{R_H^2}$.

The first condition, SE1 of the definition was that just discussed, is the main constraint of slow evolution. The second restricts the foliation so that the horizon is slowly evolving with respect to $\mathcal{V}^a$. The final two should also be true if the physical fields are also to be slowly evolving. They are essentially secondary properties that ensure conditions of the spacetime are not too extreme. Details of these are available in [1, 14].

3.7 Thermodynamics

Given these constraints for slowly evolving horizons, the thermodynamics laws are found. For the case of the classical black hole thermodynamic laws from Section 1.2 (and the more general isolated black hole thermodynamic laws [26, 23]) it is possible to define the laws in terms of the surface gravity and the angular momentum. In the same way, here generalizations of surface gravity and angular momentum are required to formulate the laws. They can be defined in terms of

$$\omega_a := -n_b \underleftarrow{\nabla}_a \mathcal{V}^b \tag{3.9}$$

as

$$\kappa_v := \mathcal{V}^a \omega_a \tag{3.10}$$

and

$$J_\varphi := \frac{1}{8\pi} \int_{H_v} d^2 \sqrt{\tilde{q}} \varphi^a \tilde{\omega}_a, \tag{3.11}$$

where φ^a is a vector field tangent to cross-sections of the horizon and $\tilde{\omega}_a$ is ω_a pulled back to H_v.

From Einstein's equations come a constraint law with terms that contain information about the flux of energy across the horizon. Two things are true of this equation: it is lengthy, as expressed in [1]; and the details of its origin are non-trivial, and can be found in [14]. To extract energy flux information in a form that resembles the laws of classical black hole thermodynamics and classical thermodynamic laws requires applying the conditions newly assigned to a slowly evolving horizon.

Zeroth Law

The first law concerning the surface gravity is

$$\kappa = \kappa^{(0)} + \epsilon\kappa^{(1)}, \tag{3.12}$$

where $\kappa^{(0)}$ is a constant. In words, the surface gravity is constant to first order. It qualifies as the zeroth law because the surface gravity is only truly constant when the black hole is in equilibrium. A black hole in equilibrium is one that is isolated. An isolated black hole has $\epsilon = 0$ which leaves (3.12) as $\kappa = \kappa^{(0)} = \text{const}$. It turns out that if the horizon satisfies certain genericity conditions for a particular choice of $\mathcal{V}^a$, then the scalings can be chosen so that the surface gravity takes the form of

$$\kappa = \kappa^{(0)} = \frac{1}{2r}, \tag{3.13}$$

which is the standard form for Schwarzschild.

First Law

Again from the diffeomorphism constraint of [14] one can derive an energy flux relation for a non-rotation horizon. From exploiting the slow evolution constraints, comes the desired law. The significant first law terms remaining are to second order in ϵ;

$$\frac{1}{8\pi G}\kappa^{(0)}\dot{a}_H = \dot{E} = \int_{H_v} d^2x\sqrt{\tilde{q}_{ab}}\left[T_{ab}\ell^a\ell^b + \frac{1}{8\pi G}\left|\sigma^{(\ell)}\right|^2\right]. \tag{3.14}$$

This relation expresses the fact that the flux of energy across the horizon comes in the form of matter flux (first term) and the gravitation radiation flux (second term).

This law can be written in the form resembling the classical black hole laws by incorporating the definition of angular momentum if φ^a is a symmetry, or at least an approximate one, of the horizon. For completeness purposes only the following version of the law is expressed here;

$$\dot{E} = \frac{1}{8\pi G}\kappa^{(0)}\dot{a} + \Omega \dot{J}_\varphi, \tag{3.15}$$

where Ω is an angular momentum. Though this version of the first law is very simple, it is cogent to use Equation 3.14 since all examples considered below will turn out to be at least perturbatively spherical.

Second Law

The two main laws of relevance are the zeroth and first. The second law is not new and is the same as for the trapping horizon. Since the slowly evolving horizon is a region of a FOTH, Equation 3.7 and the properties of C from page 58 for trapping horizons also hold for SEHs. This equation, along with the relation describing how the area element changes Equation 3.8, gives the second law. According to Equation 3.8 the relative rate of change of the area element is equal to $-C\theta_{(n)}$. It is known from TH2 that $\theta_{(n)} < 0$. Thus the minus sign and negative value of $\theta_{(n)}$ will cancel, leaving the sign of $-C\theta_{(n)}$ determined by C. Thus it immediately follows from Equation 3.7 that the relative rate of change of the area element of the FOTH must be greater than or equal to zero. This statement is the second law of black hole thermodynamics.

3.8 Summary

Various definitions were considered here for the boundary of a “black hole”. The goal was to find one that was local and physical that could represent the equilibrium and non-equilibrium states. The trapping horizon, specifically the future outer trapping horizon, was the best candidate for satisfying the ultimate goal of finding a quasi-local thermodynamical laws of black hole horizons. The FOTH allowed the formulation of a way to characterize the black hole with a trapping horizon being quasi-local. The characteristics of the slowly evolving horizon through the use of Einstein's equation resulted in thermodynamic laws that are truly analogous to those of classical thermodynamics. These thermodynamic laws are then ready for testing in particular physical examples.

In the following chapter, it is the intention to explore the zeroth and second laws for three particular spacetimes. For the formulation of the slowly evolving to be upheld the conditions expressed above for slow evolution should lead to the zeroth and first laws of the respective forms of (3.12) and (3.14). Again, while there are two more laws of black hole thermodynamics the latter two will not be discussed any further. The second law is already 'dynamic' in general and does not require a slowly evolving horizon for validation. The third law is not of interest in this thesis.

Chapter 4

Physical Examples

Testing the slowly evolving conditions of Section 3.6 and the thermodynamic laws of Section 1.1 for particular physical spacetime examples is the main work of this thesis. These conditions and laws are considered in an abridged form in [1] and then in more detail in [14]. However actual working examples are still necessary for their validation. This chapter will examine spacetimes which one would intuitively expect to be slowly evolving and test to see if they meet the conditions.

Three examples were selected. The first spacetime is the Vaidya spacetime in Eddington-Finkelstein coordinates. Vaidya describes a spherical black hole with in-falling null dust. The second spacetime is Tolman-Bondi; a spacetime of spherical collapsing timelike dust. The coordinates used are regular Gaussian coordinates with r as a coordinate that co-moves with a given dust shell. Rounding out the examples is a spacetime in which a tidally distorted black hole is present. An advanced time coordinate system is used along with a formalism of irreducible tidal fields.

4.1 The Calculations

All of the examples considered here are done in similar fashion. To consider them comprehensively, the calculations first involve finding the trapping horizon of the given spacetime. Once a trapping horizon is located, some quantities on the surface must be calculated before the conditions for slow evolution are examined as laid out in Section 3.6.

In general the details of the procedure will be the same for all examples. Before proceeding with calculating the geometrical quantities, it is necessary to find the most general null normals; outward ℓ_a and inward n_a. In each case it is assumed that the covariant forms of the null normals have no non-spherical components. That is, they have only time and radial components. By convention, they are also cross-normalized leaving only one degree of freedom associated with the null normals. This is a rescaling factor which is a general function of the coordinates. A discussion on the choice of rescaling factor will follow.

Next, it is necessary to find the induced metric $\tilde{q}_{ab}$ on the spacelike two-surface that can be described by the null normals. With the induced metric and the null normals it is possible to calculate both the outward and inward null expansions; $\theta_{(\ell)}$ and $\theta_{(n)}$ respectively. The outward null expansion is used to identify the trapping horizon. These surfaces will have zero outward null expansion and using this fact a relation between the mass function and the radial measure at the horizon (e.g. $r = 2m(v)$) can be established.

To test a trapping horizon to see if it has slowly evolving conditions it is necessary to find the tangent vector 'up' the horizon $\mathcal{V}^a$. This vector provides a way to find the scalar field C, the expansion parameter from Section 3.6, along the horizon: using the fact that the horizon must evolve in a way such that it remains trapping, it is possible to define C in terms of the physical properties of the spacetime. Specifically the outward null expansion should vanish everywhere on the horizon, and so

$$\pounds_{\mathcal{V}}\theta_{(\ell)} = 0. \tag{4.1}$$

Since it is defined in terms of C, by simply solving for the scalar C, the true form of the vector $\mathcal{V}^a$ is obtained,

$$C = \frac{\pounds_{\ell}\theta_{(\ell)}}{\pounds_{n}\theta_{(\ell)}}. \tag{4.2}$$

Imposing slowly evolving Condition 1 of page 60 gives a physical restriction associated with slow evolution in terms of the particular physical parameters of the given spacetime.

Now to reduce the remaining gauge freedom associated with the null vectors one can make use of the form of the inward null expansion of the surface. The freedom is associated with the fact that the normals are null and the length of the vector won't change even if the "size" of the vector changes. That is, if the vector is multiplied by some factor, the length does not change. An important point to note about this gauge freedom and the slowly evolving horizon conditions is that the gauge freedom does not affect the main slow evolution condition SE1 but does effect SE2. This gauge choice is tied directly with SE2. It will become apparent in the upcoming calculations that the choice of the rescaling factor is related to the vector $\mathcal{V}^a$, which in turn is related to the choice of the expansion parameter, C. Thus when restricting the foliation parameter, as in Condition 2, it comes down to restricting C which is ultimately related to the size or rescaling of the null vectors ℓ_a and n_a. It is not surprising then that the Condition 2 ultimately involves restricting the rescaling factor between the null vectors.

With the restriction it is possible to test the specific zeroth and first laws. Calculating the surface gravity tests the zeroth law. To explicitly write the first law, the stress energy term and the shear term must be evaluated. The stress energy tensor must be contracted with the outward null normal and the shear tensor must be calculated.

Some additional work is required. For example, in the tidally distorted case, work is needed in simplifying things using the spherical harmonic tensors. In order to integrate the shear tensor over the horizon it will be necessary to write the frame tidal fields in terms of the harmonic tidal fields. Also, extra simplifications are made throughout the Tolman-Bondi example using the Einstein equation and its constraint to evaluate derivatives of the area radius, $R(t, r)$.

4.2 Vaidya

The Vaidya spacetime is a solution to Einstein's equation that represents the physical situation of spherically symmetric null dust [27, 28]. The dust is imploding or infalling and uses the advanced time coordinate v.

4.2.1 Vaidya Spacetime

The solution is,

$$ds^2 = \left(-1 + 2\,\frac{m\,(v)}{r}\right) dv^2 + 2\,dv2vr + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{4.3}$$

where $m(v)$ is the mass. This mass term varies with the advanced time coordinate v. The advanced time coordinate appears since the metric of [4.3] is written as the Schwarzschild metric in Eddington-Finkelstein coordinates. The difference here from the Schwarzschild is the variable mass function that appears in front of the dv^2 term of the line element.

The Einstein tensor constructed from the metric has only one non-zero component,

$$G_{vv} = \frac{2\dot{m}(v)}{r^2}. \tag{4.4}$$

This component depends on the derivative with respect to the advanced time coordinate (represented by the overdot [1]) of the mass function. Accordingly, the stress-energy tensor has the form,

$$T_{\alpha\beta} = \frac{\dot{m}(v)}{4\pi r^2} n_\alpha n_\beta, \tag{4.5}$$

where $\dot{m}(v)$ is the derivative with respect v and the energy density of the fluid is $\rho = \dot{m}(v)/4\pi r^2$.

In these coordinates the generalized cross-normalized radial null normals to spherical $r =$ const., $v =$ const., two-surfaces are,

$$\ell_a = \left[\frac{1}{2}\left(\frac{r - 2\,m\,(v)}{\alpha\,(v,r)\,r}\right), -\frac{1}{\alpha\,(v,r)}, 0, 0\right] \tag{4.6}$$

and

$$n_a = [\alpha\,(v,r)\,, 0, 0, 0]. \tag{4.7}$$

The reader is encouraged to check that these normals are both null, $\ell_a \ell^a = 0$, $n_a n^a = 0$ and cross-normalized to -1, $\ell_a n^a = -1$. These normals also retain all possible generality outside of the obvious constraints of being null, cross-normalized and radial (no spherical components).

With these null normals at hand it is possible to find the null expansions of the two-surfaces. The induced two-metric is $\tilde{q}^{ab} = \mathrm{diag}[0, 0, 1/r^2, 1/r^2 \sin^2\theta]$. The expansion of the two-spheres is then found and used to determine the location of the trapping horizon:

$$\theta_{(\ell)} = \frac{r - 2m(v)}{\alpha(v,r) r^2}. \tag{4.8}$$

[1] The overdot will represent a derivative with respect to various coordinates. In Chapter 3 the overdot referred to a general foliation parameter. The coordinate represented by an overdot in each of these three sections is the foliation parameter.

Because the trapping horizon's outward null expansion is zero, the horizon can be located by solving $\theta_{(\ell)} = 0$. Thus the trapping horizon is located at $r = R_H = 2m(v)$.

Trapping horizons also must satisfy properties FOTH2 and FOTH3 of page 55 both of which involve $\theta_{(n)}$.

$$\theta_{(n)} = \frac{2\alpha(v,r)}{r} \tag{4.9}$$

At this point the gauge freedom can be eliminated by letting $\alpha(v,r) = -1$. It is chosen in this way to leave the expansion $\theta_{(n)}$ equal to its Schwarzschild value.

$$\theta_{(n)} = -\frac{2}{r} \tag{4.10}$$

With this gauge choice property FOTH2 for trapping horizons is satisfied. Property 3 is also satisfied since [4.9] gives the Lie derivative of the inward null expansion along the outward null vector as $(-2/r)(\partial m(v)/\partial v) < 0$.

Before investigating the slowly evolving conditions, the vector form of $\mathcal{V}^a$ must be found,

$$\mathcal{V}^a = \left[-\frac{1}{\alpha\,(v,r)}, -\frac{1}{2}\left(\frac{r - 2m(v) + 2CF^2\,(v,r)\,r}{\alpha\,(v,r)\,r}\right), 0, 0\right]. \tag{4.11}$$

Using equation 4.2 the scalar C is defined;

$$C = \frac{2\dot{m}(v)}{\alpha^2(v,r)}, \tag{4.12}$$

where again $\dot{m}(v)$ means the derivative with respect to v. Determination of this scalar gives a definite form to the vector $\mathcal{V}^a$ in terms of the mass function $m(v)$. Inherent in $\mathcal{V}^a$ is a description of how the horizon evolves in a Vaidya spacetime. With this geometrical object defined, the slowly evolving conditions can be examined.

The scalar field C associated with the vector pointing parallel to the horizon and along with property SE1 on page 60 provide the physical condition for slow evolution through,

$$C\theta_{(n)}^2 = 2\frac{\dot{m}(v)}{m^2(v)} \leq \frac{\epsilon^2}{R_H^2}, \tag{4.13}$$

which gives,

$$\dot{m}(v) \leq \frac{\epsilon^2}{8} \Leftrightarrow \rho \leq \frac{\epsilon^2}{A}, \tag{4.14}$$

where $A = 4\pi R_H$. Notice that this condition does not contain an $\alpha(v,r)$. It is the first sign that the slowly evolving conditions constructed in [1] the necessary properties. Property SE1 is supposed to be invariant of rescalings and is according to (4.14). What is more, this is a physical statement in that it says something about the physical properties of the Vaidya spacetime. The rate of change of the mass $m(v)$ with time coordinate v has to be small to second order. It also says that the horizon is slowly evolving if the product of the energy density of the dust and the area is small to second order.

The other conditions follow from here. With (4.14), C becomes of order ϵ^2;

$$C \sim \frac{\epsilon^2}{4}; \tag{4.15}$$

that is with $\alpha(v,r) = -1$. Now it is possible to evaluate the size of the vector $\mathcal{V}^a$;

$$|\mathcal{V}| = \sqrt{2C} = \sqrt{2\,(2\dot{m}(v))} \approx \sqrt{4\frac{\epsilon^2}{8}} \sim \epsilon \tag{4.16}$$

This satisfies SE2. It is important to note that this condition would not be satisfied unless the function $\alpha(v,r)$ was equal to -1, or at least some negative value very close to it ofcourse. While this choice was initially made as a choice about the form of $\theta_{(n)}$ it is now apparent that it is a result of Condition 2 of slow evolution.

SE3 has two components both of which are satisfied here. Here the ω_a associated with the horizon is,

$$\omega_a = \left[\frac{1}{4m(v)}, 0, 0, 0\right] \tag{4.17}$$

It follows from the form of ω_a that $\tilde{\omega}_a = 0$. As it turns out using (4.17) without the rescaling freedom removed (i.e. not choosing the desired value for rescaling factor) this condition is satisfied independently of the choice of $\alpha(v,r)$. Trivially it follows that the Lie derivative goes to zero,

$$\pounds_{\mathcal{V}}\tilde{\omega} = 0 \leq \frac{\epsilon^2}{R_H^2}, \tag{4.18}$$

and the first part of condition 3 is satisfied. As for the second condition;

$$\pounds_{\mathcal{V}}\theta_{(n)} = \frac{2C}{R_H^2}. \tag{4.19}$$

Using (4.15) from SE1 simplifies this expression;

$$\pounds_{\mathcal{V}}\theta_{(n)} = \frac{\epsilon^2}{2R_H^2}. \tag{4.20}$$

Thus part two of the condition is satisfied.

Considering SE4 involves finding some particular geometric quantities of the Vaidya spacetime. Firstly the Ricci scalar associated with the induced metric $\tilde{R}$ is considered. It is found to be $4m(v)/r^3$. Thus on the horizon it is $\tilde{R} = 2/R_H^2$. Secondly, the square of the vector ω_a is considered. On the horizon, with $\alpha(v,r) = -1$ gives $|\widetilde{\omega}|^2 = 0$. The shear tensor of the inward null normal is zero; $|\sigma^{(n)}|^2 = 0$. Using the stress-energy tensor it is possible to calculate the stress energy that would be measured by an observer in the Vaidya spacetime $T_{ab}n^a n^b = 0$. All these quantities are zero and thus obviously less than $1/R_H^2$, meeting the requirements of Condition 4.

Since the conditions are satisfied, the zeroth and first laws should hold. The true test is that, given the fact the slow evolution has been established, the surface gravity must be constant on the horizon. Using the equation from section 3.6 the surface gravity is found;

$$\kappa_{\mathcal{V}} = \frac{m(v)}{R_H^2} = \frac{1}{2R_H}. \tag{4.21}$$

This is of course constant on the horizon at a given v since it depends only on the horizon radius. Also, the surface gravity must evolve slowly along the horizon;

$$|\dot{\kappa}_v| = \frac{1}{2}\frac{\dot{r}}{r^2}.$$

Using this equation it is possible to evaluate the extent to which the surface gravity changes as the horizon evolves; the condition contained in (4.14) simplifies this to;

$$|\dot{\kappa}_v| \leq \frac{\epsilon^2}{8r^2}.$$

Thus the rate of change of κ_v along the horizon is small and, more specifically, is a second order quantity. This is the expected form of the surface gravity. It is the zeroth law.

It is relevant to check that the first law holds. The first law in equation (3.14) requires evaluation of the R.H.S. and the L.H.S. The left hand side uses the explicit form of the surface gravity κ;

$$\frac{\kappa \dot{A}}{8\pi} = \frac{1}{8\pi}\frac{1}{2r}\frac{\partial(4\pi r^2)}{\partial v} = \frac{\dot{r}}{2}. \tag{4.22}$$

The right hand side of the equation is also simplified greatly since here the spacetime is spherically symmetric. As a result, the integration over the area element gives the area as follows:

$$\int_{H_v} d^2x \sqrt{\tilde{q}} \left[T^{ab}\ell^a\ell^b + |\sigma^{(\ell)}|^2\right] = AT^{ab}\ell^a\ell^b = \frac{1}{4}\frac{A\left(\frac{\partial}{\partial v}m(v)\right)}{\pi r^2} = \frac{\dot{r}}{2}. \tag{4.23}$$

The R.H.S. is equivalent to the L.H.S. In this case the first law holds exactly.

4.3 Infalling Timelike Dust Shells

The physical setup considered here is the gravitational collapse of pressureless dust clouds on to a black hole. Such matter has an associated stress energy tensor of the form,

$$T_{ab} = \rho(t,r)u_a u_b = \rho\delta^t_a\delta^t_b, \tag{4.24}$$

where u_a is the 4-velocity of dust. The coordinate system used here is one with a radial coordinate r that is comoving with the collapsing shells.

The collapse here is similar to that of Oppenheimer-Snyder [29]. However, in the Tolman-Bondi collapse the energy density of the dust is not assumed to be radially homogeneous. The collapse was first described and a metric found by Tolman [30] and Bondi [31].

4.3.1 Tolman-Bondi Metric

The line element that gives the above stress energy tensor is the Tolman-Bondi metric. Using some basic assumptions listed in [30] the form of the metric can be found.

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + Y^2(t,r)d\Omega^2$$

It can be shown that by substitution of this form of the metric back into Einstein's equation $X(t,r) = \frac{1}{W(r)}\frac{\partial Y}{\partial r}$. This is shown in [31] along with the fact that an appropriate choice for the arbitrary function $W(r)$ is $W^2(r) \equiv 1 - k(r)^2$. It can also be reasoned that for all intents and purposes the function $Y(t,r)$ is a distance and is in fact the areal radius; thus the more appropriate choice of label $Y(t,r) = R(t,r)$.

The line element can now be written in terms of these functions, and following [32],

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1-k(r)}dr^2 + R(t,r)^2 d\Omega^2. \tag{4.25}$$

It is also useful to define another function;

$$m(t,r) \equiv \frac{1}{2}R(t,r)\left(\dot{R}^2 + k(r)\right),$$

[2] [31] uses $k(r) = -2E(r)$

which will obtain physical meaning once Einstein's equation is explored. The Einstein equation in terms of these auxillary functions; $m(t,r)$ and $k(r)$ can now be expressed. The general Einstein equations produce the following equations;

$$\dot{R}^2(t,r) = \frac{2m(r)}{R(t,r)} - k(r), \tag{4.26}$$

$$\dot{k} = 0, \tag{4.27}$$

and

$$\dot{m} = 0 \tag{4.28}$$

with the constraint

$$m' = 4\pi R^2(t,r)R'(t,r)T_{tt}, \tag{4.29}$$

where $T_{tt} := \rho$.

The metric and Einstein equations are fully defined in terms of the auxiliary functions

$$m(r) = 4\pi \int_{r_0}^{r} R^2(0,\tilde{r})R'(0,\tilde{r})\rho(0,\tilde{r})d\tilde{r}, \tag{4.30}$$

and

$$k(r) = \frac{2m(r)}{r} - \dot{R}^2(t,r). \tag{4.31}$$

These functions have physical interpretations. $m(r)$ is a mass function describing the mass inside a given spherical shell of radius r. $k(r)$ tells whether the system is gravitationally bound. The system is bound if $k(r) > 0$, spacelike if $k(r) < 1$.

When carrying out the calculations for the Tolman-Bondi spacetime, it is necessary to simplify each object by using the Einstein equation of (4.26) and (4.29) to evaluate $\dot{R}(t,r)$ and $R'(t,r)$ respectively.

4.3.2 SE Horizon

Finding the null normals using the metric in equation (4.25) gives,

$$n^a = \frac{1}{2}\frac{1}{\alpha(t,r)}\left[1, -\frac{\sqrt{1-k(r)}}{R(t,r)'}, 0, 0\right], \tag{4.32}$$

and

$$\ell^a = \alpha(t,r)\left[1, \frac{\sqrt{1-k(r)}}{R(t,r)'}, 0, 0\right]. \tag{4.33}$$

Upon calculation of the expansions, one can find the location of a trapping horizon and make a gauge choice. First the induced metric is diag$[1/R^2(t,r), 1/R^2(t,r)\sin^2\theta]$. On the other hand, the outward null expansion is of the form,

$$\theta_{(\ell)} = \frac{2\left(-\sqrt{2\frac{m(r)}{R(t,r)} - k(r)} + \sqrt{1-k(r)}\right)\alpha(t,r)}{R(t,r)}. \tag{4.34}$$

It can easily be seen that $\theta_{(\ell)} = 0$ when $R(t,r) = R_H = 2m(r)$ which defines a trapping horizon.

On the other hand, the form of the inward null expansion is

$$\theta_{(n)} = \frac{\frac{\partial}{\partial t}R(t,r) - \sqrt{1-k(r)}}{R(t,r)\,\alpha(t,r)},$$

It is now possible to use the freedom in the scaling factor to set $\theta_{(n)}$ to $\frac{-2}{R(t,r)}$. Then using $\theta_{(\ell)} = 0$ along with Einstein's equation $\theta_{(n)}$ simplifies to

$$\theta_{(n)} = \frac{-2\sqrt{1-k(r)}}{\alpha(t,r)R(t,r)}. \tag{4.35}$$

Thus the preferred gauge choice will give the value for $\alpha(t,r)$;

$$\alpha(t,r) = \sqrt{1-k(r)}.$$

For most of the calculated results reported after this point, the determined substitution for the rescaling factor will already be invoked for the results reported. Some other substitutions will already be made as well, like that for the trapping horizon condition, the Einstein equation and its associated constraint.

Next, the form of the vector along the horizon is sought;

$$\mathcal{V}^a = \ell^a - Cn^a = \frac{1}{2}\left[\frac{2(1-k(r)) - C}{\sqrt{1-k(r)}}, \frac{(2(1-k(r)) + C)\,A\rho}{m'(r)}, 0, 0\right], \tag{4.36}$$

where again $A = 4\pi R_H^2$. Having the form of $\mathcal{V}^a$ allows the calculation of C. Using (3.7) gives C in terms of the physical parameters of the horizon;

$$C = \frac{2(1-k(r))A\rho}{1-A\rho}. \tag{4.37}$$

This is a measure of the evolution of the horizon in terms of the characteristics of the Tolman-Bondi spacetime. It is now possible to check to see that the SE conditions are satisfied. SE1 regarding the invariance of slow expansion has a left hand side

$$C\theta_{(n)}^2 = \frac{32\pi(1-k(r))\rho A}{1-A\rho}, \tag{4.38}$$

thus such a horizon is slowly evolving if,

$$C\theta_{(n)}^2 \leq \frac{\epsilon^2}{R^2} \Rightarrow A\rho \approx \frac{\epsilon^2}{32\pi(1-k(r)) + \epsilon^2} \approx \epsilon^2. \tag{4.39}$$

Selecting $0 < k(r) < 1$, as before, equation (4.39) produces the relation

$$\rho \approx \frac{\epsilon^2}{A}. \tag{4.40}$$

SE2 is the restriction of foliation with respect to $\mathcal{V}^a$. From SE1 of the Tolman-Bondi spacetime, it is possible to learn the limit on the expansion parameter,

$$C \approx \frac{\epsilon^2}{4}. \tag{4.41}$$

Condition 2 is easily achieved with the value of C,

$$|\mathcal{V}| = \sqrt{2C} \approx \sqrt{2\frac{\epsilon^2}{4}} \sim \epsilon.$$

For condition SE3, there are two components both of which are satisfied here as well. The vector $\widetilde{\omega}$ on the surface goes to zero. That leaves $|\pounds_{\mathcal{V}}\widetilde{\omega}_a|$ as zero and less than ϵ/R^2. As for the second part of the condition,

$$\pounds_{\mathcal{V}}\theta_{(n)} = \frac{4(1-k(r))A\rho}{R^2(t,r)(1-A\rho)}.$$

The Lie derivative simplifies:

$$\pounds_{\mathcal{V}}\theta_{(n)} = \frac{1}{2}\frac{\epsilon^2}{R_H^2}.$$

Together these two relations satisfy those properties set out in condition SE3.

The Ricci scalar on the surface $\tilde{\mathcal{R}}$ is the first geometric object of consideration in SE4 and for this case it has the relation,

$$\tilde{\mathcal{R}} = \frac{2}{R_H^2}.$$

$|\omega|^2$ is zero. The shear tensor associated with the inward null vector, $|\sigma^{(n)}|$ also goes to zero. Finally the value of $T_{ab}n^a n^b$ goes to zero. These combine to fully satisfy condition SE4.

With the slowly evolving conditions met, the zeroth and first should hold. They are checked below. The form of the surface gravity is required to evaluate both laws. The surface gravity is,

$$\kappa_{\mathcal{V}} = \frac{1}{4m(r)} + \frac{C}{8m(r)(-1+k(r))}$$

where the value for C can be substituted from equation (4.41). Invoking the slowly evolving condition shows that the surface gravity to first order is

$$\kappa_{\mathcal{V}} \approx \frac{1}{2R_H},$$

as predicted by the zeroth law of slowly evolving horizons. This is the point in the calculation that first validates the SE regime and the thermodynamic laws for the Tolman-Bondi case.

Therefore the first law can be evaluated using $\kappa_{\mathcal{V}}$ from equation (4.3.2). The L.H.S. of the first law is,

$$\frac{\kappa \dot{A}}{8\pi} = \frac{1}{32}\frac{A(\dot{v})}{m(r)\pi} = \frac{\dot{R}_H}{2}. \tag{4.42}$$

As for the R.H.S. of the first law, the second term goes to zero since $|\sigma^{(\ell)}|^2$ goes to zero and in the first term $T^{ab}\ell^a\ell^b = \rho\,(1-k(r))$ thus,

$$\int_{H_v} d^2x\sqrt{\tilde{q}}\left[T^{ab}\ell^a\ell^b + |\sigma^{(\ell)}|^2\right] = A(1-k(r))\rho. \tag{4.43}$$

Using the Einstein equation (4.28) the R.H.S becomes

$$\int_{H_v} d^2x\sqrt{\tilde{q}}\left[T^{ab}\ell^a\ell^b + |\sigma^{(\ell)}|^2\right] \approx \frac{\dot{R}_H^2}{2}. \tag{4.44}$$

Since on the horizon ($r = 2m(r)$) and Equation (4.28) of Einstein's equations hold, both the L.H.S. and R.H.S. both go to zero. This satisfies the first law.

4.4 Tidally Distorted Black hole

Here a black hole that is distorted tidally by its surrounding spacetime is considered. It is the Weyl tensor of the external spacetime that is responsible for the distortion. The related radius of curvature of the spacetime is $\mathtt{R}$ and the black hole has mass M. Another assumption about the spacetime is that there is no matter in the area surrounding the black hole. The coordinate system used to describe the spacetime and define the metric is an advanced time coordinate system.

A difference in this example with the Tolman-Bondi and Vaidya spacetimes is that a physical restriction is already in place on the spacetime. Instead of necessarily using the slowly evolving conditions for the spacetime to find out something about the physical parameters, examples of which here are M and $\mathtt{R}$, a restriction is already assumed:

$$\frac{M}{\mathtt{R}} \ll 1. \tag{4.45}$$

A check of the slowly evolving condition will either indicate that this scenario and the associated physical restriction, ensures slow evolution or will suggest a modification to the restriction. As it stands, there will be an interest only in retaining the calculated objects to first order in $\frac{M}{\mathtt{R}}$.

In this spacetime it is possible to relate the Weyl tensor to symmetric, tracefree frame tensors. These tensors are the tidal fields $\mathcal{E}_{ab} = C_{a0b0}$, and $\mathcal{B}_{ab} = \frac{1}{2}\varepsilon_a^{pq}C_{pqb0}$, where ε_{abc} is the three dimensional permutation tensor. These frame components along with the derivatives of the Weyl tensor, which can also be written in terms of the symmetric, tracefree frame tensors, $\mathcal{E}_{abc} = \frac{1}{3}\left(C_{a0b0}|_c + C_{c0a0}|_b + C_{b0c0}|a\right)$ and $\mathcal{B}_{abc} = \frac{1}{8}\left(\varepsilon_a^{pq}C_{pqbo}|_c + \varepsilon_b^{pq}C_{pqa0}|_b + \varepsilon_c^{pq}C_{pqc0}|_a\right)$, allow for a definition of the convenient harmonic components of the tidal fields $\mathcal{E}^q$, $\mathcal{E}_A^q$, $\mathcal{E}_{AB}^q$, $\mathcal{B}_A^q$, $\mathcal{B}_{AB}^q$. (A,B represent indicies that my have the value of the coordinates θ and ϕ which are also can be labelled as 3 and 4 respectively)

This situation was studied by Poisson after Alvi's use of a technique for matching two perturbed Schwarzschild metrics was established. Poisson's version of the metric betters Alvi's by an order of magnitude. The metric displayed in equation (4.46), while very lengthy, admits relatively simple null normals and geometric objects. The metric is defined in Table 4.1, however, most of the components will go to zero on the horizon. The function $f(r)$ is defined in the usual way as a simple function of r as $1 - \frac{2M}{r}$.

$$
\begin{aligned}
g_{vv} &= -f\left(r\right) - r^2 e_1\left(r\right)\,\mathcal{E}^q\left(v,\theta,\phi\right) + 1/3\,r^3 e_2\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q\left(v,\theta,\phi\right) - \\
&\quad 1/3\,r^3 e_3\left(r\right)\,\mathcal{E}^o\left(v,\theta,\phi\right) \\
g_{vr} &= 1 \\
g_{v\theta} &= -2/3\,r^3\left(e_4\left(r\right)\,\mathcal{E}^q_3\left(v,\theta,\phi\right) - b_4\left(r\right)\,\mathcal{B}^q_3\left(v,\theta,\phi\right)\right) + \\
&\quad 1/3\,r^4\left(e_5\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q_3\left(v,\theta,\phi\right) - b_5\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{B}^q_3\left(v,\theta,\phi\right)\right) - \\
&\quad 1/4\,r^4\left(e_6\left(r\right)\,\mathcal{E}^o_3\left(v,\theta,\phi\right) - b_6\left(r\right)\,\mathcal{B}^o_3\left(v,\theta,\phi\right)\right) \\
g_{v\phi} &= -2/3\,r^3\left(e_4\left(r\right)\,\mathcal{E}^q_4\left(v,\theta,\phi\right) - b_4\left(r\right)\,\mathcal{B}^q_4\left(v,\theta,\phi\right)\right) + \\
&\quad 1/3\,r^4\left(e_5\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q_4\left(v,\theta,\phi\right) - b_5\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{B}^q_4\left(v,\theta,\phi\right)\right) - \\
&\quad 1/4\,r^4\left(e_6\left(r\right)\,\mathcal{E}^o_4\left(v,\theta,\phi\right) - b_6\left(r\right)\,\mathcal{B}^o_4\left(v,\theta,\phi\right)\right) \\
g_{\theta\phi} &= -1/3\,r^4\left(e_7\left(r\right)\,\mathcal{E}^q_{34}\left(v,\theta,\phi\right) - b_7\left(r\right)\,\mathcal{B}^q_{34}\left(v,\theta,\phi\right)\right) + \\
&\quad \tfrac{5}{18}\,r^5\left(e_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q_{34}\left(v,\theta,\phi\right) - b_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{B}^q_{34}\left(v,\theta,\phi\right)\right) - \\
&\quad 1/6\,r^4\left(e9\left(r\right)\,\mathcal{E}^o_{34}\left(v,\theta,\phi\right) - b_9\left(r\right)\,\mathcal{B}^o_{34}\left(v,\theta,\phi\right)\right) \\
g_{\theta\theta} &= r^2 - 1/3\,r^4\left(e_7\left(r\right)\,\mathcal{E}^q_{33}\left(v,\theta,\phi\right) - b_7\left(r\right)\,\mathcal{B}^q_{33}\left(v,\theta,\phi\right)\right) + \\
&\quad \tfrac{5}{18}\,r^5\left(e_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}\left(v,\theta,\phi\right) - b_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}\left(v,\theta,\phi\right)\right) - \\
&\quad 1/6\,r^4\left(e_9\left(r\right)\,\mathcal{E}^o_{33}\left(v,\theta,\phi\right) - b_9\left(r\right)\,\mathcal{B}^o_{33}\left(v,\theta,\phi\right)\right) \\
g_{\phi\phi} &= \sin^2(\theta) r^2 + \\
&\quad \tfrac{1}{3}\,r^4\sin^2\left(\theta\right)\left(e_7\left(r\right)\,\mathcal{E}^q_{33}\left(v,\theta,\phi\right) - b_7\left(r\right)\,\mathcal{B}^q_{33}\left(v,\theta,\phi\right)\right) - \\
&\quad \tfrac{5}{18}\,r^5\sin^2\left(\theta\right)\left(e_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}\left(v,\theta,\phi\right) - b_8\left(r\right)\,\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}\left(v,\theta,\phi\right)\right) \\
&\quad + \tfrac{1}{6}\,r^4\left(\sin\left(\theta\right)\right)^2\left(e_9\left(r\right)\,\mathcal{E}^o_{33}\left(v,\theta,\phi\right) - b9\left(r\right)\,\mathcal{B}^o_{33}\left(v,\theta,\phi\right)\right)
\end{aligned}
\tag{4.46}
$$

Here, the first step is guessing that the $r =$ const., $v =$ const. two surfaces will foliate the slowly evolving horizon. The calculation will explore this. Equation (4.46) is all that is needed to find the null normals ℓ_a and n_a. The general form of the null normals is of course found to be using the metric and ensuring that the normals have a magnitude of zero and that ℓ_a and n_a are cross-normalized to -1. The outward null normal is simple to first order and the only component dependent on the tidal fields is the r component.

$$
\begin{aligned}
\ell_v &= \tfrac{1}{\alpha(v,r)}\left(-\tfrac{1}{6}3f(r) + 3r^2 e_1(r)\mathcal{E}^q(v,\theta,\phi)\right. \\
&\quad \left. + r^3 e_3(r)\mathcal{E}^o(v,\theta,\phi) - r^3 e_2(r)\tfrac{\partial}{\partial v}\mathcal{E}^q(v,\theta,\phi)\right) \\
\ell_r &= \tfrac{1}{\alpha(v,r)} \\
\ell_\theta &= 0 \\
\ell_\phi &= 0
\end{aligned}
\tag{4.47}
$$

As is now the norm in these examples, the inward null normal is only the v-component that is the reciprocal of the scaling factor of the null normals;

$$
n_a = \alpha(v,r,\theta,\phi)[1,0,0,0]. \tag{4.48}
$$

The induced metric for tidally distorted spacetime is lengthy and depends on the tidal fields;

$$
\begin{aligned}
q^{\theta\theta} &= \tfrac{1}{18r^2}\left(18 + 6r^2 e_7(r)\,\mathcal{E}^q_{33}(v,\theta,\phi) - 6r^2 b_7(r)\mathcal{B}^q_{33}(v,\theta,\phi) - \right. \\
&\quad 5r^3 e_8(r)\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}(v,\theta,\phi) + 5r^3 b_8(r)\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}(v,\theta,\phi) + \\
&\quad \left. 3r^3 e_9(r)\mathcal{E}^o_{33}(v,\theta,\phi) - 3r^3 b_9(r)\mathcal{B}^o_{33}(v,\theta,\phi)\right) \\
q^{\theta\phi} = q^{\phi\theta} &= \tfrac{1}{18}\left(6e_7(r)\mathcal{E}^q_{34}(v,\theta,\phi) + 6b_7(r)\mathcal{B}^q_{34}(v,\theta,\phi) + \right. \\
&\quad 5re_8(r)\tfrac{\partial}{\partial v}\mathcal{E}^q_{34}(v,\theta,\phi) - 5rb_8(r)\tfrac{\partial}{\partial v}\mathcal{B}^q_{34}(v,\theta,\phi) - \\
&\quad \left. 3re_9(r)\mathcal{E}^o_{34}(v,\theta,\phi) + 3rb_9(r)\mathcal{B}^o_{34}(v,\theta,\phi)\right) \\
q^{\phi\phi} &= \tfrac{1}{18r^2\sin^2(\theta)}\left(18 - 6r^2 e_7(r)\mathcal{E}^q_{33}(v,\theta,\phi) + 6r^2 b_7(r)\mathcal{B}^q_{33}(v,\theta,\phi) + \right. \\
&\quad 5r^3 e_8(r)\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}(v,\theta,\phi) - 5r^3 b_8(r)\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}(v,\theta,\phi) - \\
&\quad \left. 3r^3 e_9(r)\mathcal{E}^o_{33}(v,\theta,\phi) + 3r^3 b_9(r)\mathcal{B}^o_{33}(v,\theta,\phi)\right).
\end{aligned}
\tag{4.49}
$$

The expression for the outward null expansion proves it to also be a fairly lengthy expression, so much so that it is not displayed here. However, it is straightforward to find that it vanishes to second order on the surface defined by $r = 2m$. Because the metric is also to second order, this surface is a trapping horizon.

$e_1 = f^2$
$e_2 = f[1 + \frac{1}{4x}(5 + \ln x) - \frac{3}{4x^2}(9 + 4\ln x) + \frac{7}{4x^3} + \frac{3}{4x^4}]$
$e_3 = f^2(1 - \frac{1}{2x}$
$e_4 = f$
$e_5 = f\left[1 + \frac{1}{6x}(13 + 12\ln x) - \frac{5}{2x^2} - \frac{3}{2x^3} - \frac{1}{2x^4}\right]$
$e_6 = f\left(1 - \frac{2}{3X}\right)$
$e_7 = 1 - \frac{1}{2x^2}$
$e_8 = 1 + \frac{2}{5x}(4 + 3\ln x) - \frac{9}{5x^2} - \frac{1}{x^3}(7 + 3\ln x) + \frac{3}{5x^4}$
$e_9 = f + \frac{1}{10x^3}$
$b_4 = f$
$b_5 = f\left[1 + \frac{1}{6x}(7 + 12\ln x) - \frac{3}{2x^2} - \frac{1}{2x^3} - \frac{1}{6x^4}\right]$
$b_6 = f\left(1 - \frac{2}{3x}\right)$
$b_7 = 1 - \frac{3}{2x^2}$
$b_8 = 1 + \frac{1}{5x}(5 + 6\ln x) - \frac{9}{5x^2} - \frac{1}{5x^3}(2 + 3\ln x) + \frac{1}{5x^4}$
$b_9 = f - \frac{1}{10x^3}$

Table 4.1: The radial functions define the metric where $x = r/2m$. Most of them go to zero on the trapping horizon when $r = 2M$ with the exception of $e_7 = \frac{1}{2}$, $e_9 = \frac{1}{10}$, $b_7 = -\frac{1}{2}$ and $b_9 = -\frac{1}{10}$.

Finding the inward null expansion, $\theta_{(n)}$ gives the expected result once the gauge choice is made to find the rescaling factor, $\alpha(v, r, \theta, \phi) = 1$. On $r = 2M$,

$$\theta_{(n)} = -2\frac{\alpha\,(v, r, \theta, \phi)}{r} + \mathcal{O}(\epsilon^2) \approx -\frac{2}{r}. \tag{4.50}$$

The vector $\mathcal{V}^a$ gives an expansion parameter C of zero to second order, which makes things extremely simple. Using the form of $\theta_{(\ell)}$ and $\mathcal{V}^a$, ensuring that the outward null expansion is Lie dragged along $\mathcal{V}^a$, indicates that the C must vanish.

With $C \approx 0$ to second order the slowly evolving conditions are easily satisfied. For SE1,

$$C\theta^2_{(n)} = \frac{4C\alpha^2(v, r, \theta, \phi)}{r^2} + \mathcal{O}(\epsilon^2) \approx 0. \tag{4.51}$$

The tidally distorted spacetime can invariantly be classified as slowly expanding.

Trivially, SE2 is satisfied,

$$|\mathcal{V}^a| = \sqrt{2C} \approx 0. \tag{4.52}$$

Restrictions of both quantities of SE3 are also easily satisfied. The components of the one-form $\tilde{\omega}_a$ are as follows;

$$\begin{aligned}
\tilde{\omega}_v &= 0 \\
\tilde{\omega}_r &= 0 \\
\tilde{\omega}_\theta &= \tfrac{1}{9}M^2 \big(16M\tfrac{\partial}{\partial v}\mathcal{E}^q_3(v, \theta, \phi) + 12\mathcal{E}^q_3(v, \theta, \phi) - \\
&\quad 12\mathcal{B}^q_3(v, \theta, \phi) + 3M\mathcal{E}^o_3(v, \theta, \phi) - 3M\mathcal{B}^o_3(v, \theta, \phi)\big) \\
\tilde{\omega}_\phi &= \tfrac{1}{9}M^2 \big(16M\tfrac{\partial}{\partial v}\mathcal{E}^q_4(v, \theta, \phi) + 12\mathcal{E}^q_4(v, \theta, \phi) - \\
&\quad 12\mathcal{B}^q_4(v, \theta, \phi) + 3M\mathcal{E}^o_4(v, \theta, \phi) - 3M\mathcal{B}^o_4(v, \theta, \phi)\big).
\end{aligned} \tag{4.53}$$

With the one-form, its rate of change along the horizon can be evaluated as,

$$|\pounds_{\mathcal{V}}\tilde{\omega}_a| = 0. \tag{4.54}$$

Since C goes to zero as a result of Lie dragging the outward null expansion from above,

$$|\pounds_{\mathcal{V}}\theta_{(n)}| = \frac{2C}{r^2} + \mathcal{O}(\epsilon^2) \approx 0. \tag{4.55}$$

Both are less than ϵ/r^2 as required.

The last set of conditions are also satisfied. The first condition regarding the Ricci scalar on the surface must impose some conditions on how the irreducible tidal fields behave on the surface. The other three components are, however, satisfied without any new constraints. Using (4.53) it is possible to see that $|\omega|^2$ vanishes. Also, the non-zero components of the shear tensor are;

$$\begin{aligned}
\sigma^{(n)}_{\theta\theta} &= -\tfrac{1}{36}r^3\left(22\left(\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}(v,\theta,\phi)\right)r - 12\mathcal{E}^q_{33}(v,\theta,\phi) + 12\mathcal{B}^q_{33}(v,\theta,\phi)\right. \\
&\quad \left. -18\left(\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}(v,\theta,\phi)\right)r - 3r\mathcal{E}^o_{33}(v,\theta,\phi) + 3r\mathcal{B}^o_{33}(v,\theta,\phi)\right) \\
\sigma^{(n)}_{\theta\phi} = \sigma^{(n)}_{\theta\phi} &= -\tfrac{1}{36}r^3\left(22\left(\tfrac{\partial}{\partial v}\mathcal{E}^q_{34}(v,\theta,\phi)\right)r - 12\mathcal{E}^q_{34}(v,\theta,\phi) + 12\mathcal{B}^q_{34}(v,\theta,\phi)\right. \\
&\quad \left. +18\left(\tfrac{\partial}{\partial v}\mathcal{B}^q_{34}(v,\theta,\phi)\right)r - 3r\mathcal{E}^o_{34}(v,\theta,\phi) + 3r\mathcal{B}^o_{34}(v,\theta,\phi)\right) \\
\sigma^{(n)}_{\phi\phi} &= \tfrac{1}{36}\sin(\theta)^2 r^3\left(22\left(\tfrac{\partial}{\partial v}\mathcal{E}^q_{33}(v,\theta,\phi)\right)r - 12\mathcal{E}^q_{33}(v,\theta,\phi) + 12\mathcal{B}^q_{33}(v,\theta,\phi)\right. \\
&\quad \left. -18\left(\tfrac{\partial}{\partial v}\mathcal{B}^q_{33}(v,\theta,\phi)\right)r - 3r\mathcal{E}^o_{33}(v,\theta,\phi) + 3r\mathcal{B}^o_{33}(v,\theta,\phi)\right),
\end{aligned} \tag{4.56}$$

so $|\sigma^{(n)}| = 0$. As a result, both $|\omega|^2$ and $|\sigma^{(n)}|$ are approximatly or less than $\frac{1}{R_H^2}$. Also, the double contraction of the stress energy tensor with the inward pointing null vectors, $T_{ab}n^a n^b$ reduces to zero.

Now that it is established that the horizon is slowly evolving the zeroth and first law can be checked. The surface gravity associated with the horizon for the tidally distorted spacetime is,

$$\kappa_v = \frac{1}{4M} + \frac{4}{3}M^2\left(\frac{\partial}{\partial v}\mathcal{E}^q(v,\theta,\phi)\right).$$

Therefore, since on the trapping horizon $r = 2M$, the surface gravity simplifies to

$$\kappa_v^0 = \frac{1}{2r}, \tag{4.57}$$

which is the expected result.

Expounding on this result, one can write the equation for the first law. Knowing the surface gravity makes possible the expression of the energy across the horizon. The L.H.S. of the first law is,

$$\frac{\kappa \dot{A}}{8\pi} = \frac{1}{32}\frac{\dot{A(v)}}{M\pi} = \frac{\dot{r}}{2}. \tag{4.58}$$

The overdot here indicates the derivative with respect to the foliation parameter. To evaluate R.H.S. the first term is calculated. The first term involves contracting the stress energy tensor twice with outward null expansion. $T_{ab}\ell^a\ell^b = 0$, since there is no matter. The general form simplifies to

$$\int_{H_v} d^2x\sqrt{\tilde{q}}\left[T^{ab}\ell^a\ell^b + |\sigma^{(\ell)}|^2\right] = 0 + \int_{H_v} d^2x\sqrt{\tilde{q}}|\sigma^{(\ell)}|^2.$$

Expanding the shear term is, however, fairly involved. The shear of the outward null normals has many terms mostly composed of derivatives of the irreducible tidal fields. To simplify the expression it is necessary to use the spherical-harmonic decomposition of the irreducible tidal fields as outlined in the Table II (Appendix) from [33]. Using the table it is possible, with some effort, to markedly simplify the R.H.S. of the shear expression in terms of the frame components of the tidal fields ($\mathcal{E}$). Also required is the determinant of the metric on the surface, $\sqrt{\tilde{q}} = r^2\sin(\theta)$.

$$\begin{aligned}
&M^6 \text{ order Shear Term} = \\
&\tfrac{32}{45}\,\epsilon^2 \Big(\left(\tfrac{d}{dv}\mathcal{E}_{11}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{B}_{23}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{B}_{11}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{B}_{22}(v)\right)^2 + \\
&\left(\tfrac{d}{dv}\mathcal{E}_{23}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{E}_{22}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{B}_{13}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{E}_{13}(v)\right)^2 + \\
&\left(\tfrac{d}{dv}\mathcal{B}_{12}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{E}_{12}(v)\right)^2 + \left(\tfrac{d}{dv}\mathcal{E}_{11}(v)\right)\tfrac{d}{dv}\mathcal{E}_{22}(v) + \\
&\left(\tfrac{d}{dv}\mathcal{B}_{11}(v)\right)\tfrac{d}{dv}\mathcal{B}_{22}(v)\Big)\,M^6.
\end{aligned} \tag{4.59}$$

Further simplification is allowed because the tidal fields are tracefree and their indices are raised with δ_{ab},

$$\frac{16}{45}M^6\left(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab}+\dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}\right).$$

A similar procedure can be carried out for the M^8 term giving up the final version of the R.H.S. first law;

$$\int_{H_v} d^2x\sqrt{\tilde{q}}\left[T^{ab}\ell^a\ell^b+|\sigma^{(\ell)}|^2\right]=\frac{16}{45}M^6\left(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab}+\dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}\right)+\frac{16}{4725}M^8\left(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc}+\frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\right). \tag{4.60}$$

Thus the first law is satisfied to first order since both (4.58) and (4.60) are second order. Here it is interesting that these second order quantities obtained in this calculation have provided a simple way to calculate the change in energy across the horizon,

$$\dot{E}=\frac{16}{45}M^6\left(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab}+\dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}\right)+\frac{16}{4725}M^8\left(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc}+\frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\right). \tag{4.61}$$

This first law equation says that the rate of change of the energy across the surface is equal to the change in the horizon radius to second order. This in turn equals a somewhat simple sum of dot products of frame components of the tidal fields times orders of M^6 and M^8 as laid out in (4.61).

This result matches that of [16] where here, the terms represent the physical picture at a given point in time while Poisson deals with time averaged quantities.

Chapter 5

Discussion and Conclusions

Before discussing and summarizing the results of the calculations in 5.2, a brief Review of the Motivation leading up to the calculations will first put them into perspective. Concluding the thesis will be a discussion of some Philosophy and Applications and some notes on Further Study.

5.1 Review of Motivation

There are various ways to define the boundary of a black hole. The most widely accepted ones were discussed in Chapter 3. The case for using the trapping horizon has been made in Section 3.4 whereas the following are true. The event horizon is non-local in that it requires going to spacial infinity and waiting an infinite time to define the boundary. The situation is only moderately improved by the apparent horizon since the apparent horizon is only temporally local. The trapping horizon however is quasi-local and thus more physical.

There is an important property of trapping horizons which makes it the most appropriate horizon for the work of this thesis. Unlike any other horizon discussed here, besides the apparent horizon, it encompasses equilibrium and non-equilibrium states. Using the trapping horizon formalism, one can identify the boundary of the black hole both when it is dynamic and when it is in equilibrium. Mathematically this means that the submanifold H, which is foliated by trapped surfaces, can be either null or spacelike. Respectively this corresponds to equilibrium and non-equilibrium states. The equilibrium case corresponds mathematically to the case where the shear σ_{ab} of ℓ^a and $T_{ab}l^a l^b$ go to zero across the horizon [24]. This has physical meaning and is intuitively what is expected. The fact that these two types of states are covered, coupled with the fact that it is more local than other horizon formalisms, makes this the horizon of choice for formulating a slowly evolving regime.

It should be apparent now though that it is the slowly evolving horizon that is truly physical in the same sense as in quasi-equilibrium thermodynamics. The situation of having to go from dynamic to slowly evolving was seen above (Chapter 1) to be analogous to obtaining quasi-equilibrium thermodynamics from the wider ranging thermodynamics. If a system is in equilibrium one can talk about the properties that define the state of the system. As in thermodynamics, going to a quasi-equilibrium state allows the definition of properties or state variables that describe the state of a system. The important quantity in thermodynamics is the temperature; it is the quantity used to define equilibrium. When two objects are in equilibrium with a third they have the same constant temperature. In such cases it also makes sense to talk about the state variable pressure. Quasi-equilibrium states thus allow the sate of the system to be identified by these variables. In that vein, it can be said that quasi-equilibrium thermodynamics is more *physical* than non-equilibrium thermodynamics It is in this way that for black hole thermodynamics the slowly evolving horizon is more physical than other dynamic horizon definitions. Here the important quantity is the surface gravity. With the proper conditions, a trapping horizon will be quasi-static and will a have constant temperature associated with the horizon. The properties of the black hole defined in such a way are 'measurable' and determine the state of the black hole. To that end, the properties of slowly evolving horizons were established.

Further motivation for SEH, within black hole thermodynamics, is also evident. As mentioned by Hayward there are problems with the thermodynamics described by the stationary black hole thermodynamics. Such a regime only covers thermostatics and does not describe fully a possible evolving black hole. What is more, the second law for this case is the only one that carries any generality. The zeroth and first law do not encompass a definition surface gravity and energy conservation of a black hole which passes through equilibrium states and is not stationary.

The conditions for slow evolution as outlined by Booth et al. have characteristics that serve to prevent a false representation of a slowly evolving horizon. Great care has been taken to find conditions that classify slow evolution which do not depend on anything other than what gives a physically intuitive definition of the horizon's slow evolution. Slowly Evolving horizons are invariantly characterized by a slow expansion; in particular with respect to the vector along the horizon $\mathcal{V}^a$. The inward null expansion and the vector $\widetilde{\omega}_a$ are also changing slowly along $\mathcal{V}^a$. Some other geometric quantities are also restricted to slow evolution. Those conditions have been respectively laid out in the properties on page 60. They have just been put to test in Chapter 4.

5.2 Examples of Slow Evolution

It has been the program of this thesis to illustrate examples of a few specific spacetimes and their satisfaction of the two main features of slowly evolving horizons. To be a slowly evolving spacetime, a black hole spacetime must posses appropriate physical restrictions. These must originate from the general slow evolution conditions which should be physically meaningful and intuitively expected. In addition, such a spacetime must obey the zeroth and first laws of black hole thermodynamics as predicted for the slowly evolving regime. The examples considered in Chapter 4 do validate all the slowly evolving conditions and this is self-evident from the calculations.

Vaidya

In the case of the Vaidya spacetime, the physical implications of slow evolution were discovered in (4.14). This equation says that the rate of change of the mass across the horizon is small in comparison to the area of the horizon to second order. More simply, the density of the infalling matter is small to second order compared to the area of the horizon. Physically this is instinctive.

The Vaidya surface gravity is defined in (4.21) and is a constant on the horizon. The surface gravity is also slowly evolving along the horizon. Thus a black hole of this type will be in equilibrium if all regions of the hole have the same value for this surface gravity. The first law holds since the L.H.S. of the first law equation, (4.22) is equal to the R.H.S. of (4.23). The first law itself states that the energy crossing the horizon is the rate of change of the mass along the horizon. The Vaidya example proves to be a slowly evolving spacetime where the zeroth and first laws hold.

Tolman-Bondi

Invoking the slowly evolving conditions gives a physical restriction on the Tolman-Bondi spacetime. Equation (4.40) is very similar to the equivalent version for the Vaidya spacetime. It says that the density of the infalling matter is small to second order compared with the area of the horizon. Again, this is the type of restriction that one would naturally expect.

The surface gravity associated with a slowly evolving horizon in the Tolman Bondi spacetime is expressed in §4.3.2. It is essentially the same as that from Vaidya. This provides an indicator of what equilibrium state the black hole is in at a given time measure. From this surface gravity follows the first law; (4.42) equals (4.44). The first law for both of the collapsing matter spacetimes are essentially the same.

Tidally Distorted

The physical parameters issued in the setup of the tidally distorted case turned out to be sufficient for slow evolution. That is, the fact that the mass of the black hole is much smaller than the radius of curvature of the external spacetime ensures slow evolution. This condition, which was already established in (4.45) and used in the derivation in the metric, turns out to be sufficient to allow SE1 to go to zero. Surface gravity for the tidally distorted hole is (4.57). This gives the first law as (4.61).

While the slowly evolving laws are valid in this case there is also a particular aspect of Poisson's tidally distorted black hole that make it different from the other two examples. The first law depends on the tidal fields whereas in the other two examples it depends on the area and density of a shell of dust.

A fact of the tidally distorted black hole that is worthy of discussion here is that in [33] Poisson proved to have the same thermodynamic first law as was calculated above. The first law in that case though was defined for the event horizon and in terms of time averages of the quantities like the rate of change of the mass and angular momentum. These time averages are in conjunction with the reality that the properties of the event horizon are not local. Here, the first law is in terms of the thermodynamics at s given instant in time and is more physical. Because in this case it was established that the slowly evolving regime was in effect, as was done above, the first law followed quite easily in comparison to that for the event horizon. In

[33], much effort was required to obtain the event horizon equivalent to the first law here within, where the Regge-Wheeler and Zerilli equations [34, 35] were used. While this calculation is no doubt distinct from the one above and quite useful, especially when exploring the full dynamics of the event horizon, for the case of slowly evolving regime a first law is more physical and accordingly obtained much more efficiently.

Of the slow evolution conditions it is also worth noting that the for general slowly evolving horizons the conditions are invariantly characterized.

5.2.1 Summary of Results

Reasonable physical conditions in each of these cases were satisfied. Each of the first laws were established after the surface gravity was determined. The surface gravity of each of them fits the same form. As was predicted in [1], the form for these near Schwarzschild can be written as,

$$\kappa_v^{(0)} = \frac{1}{2r_H},$$

where r_H is the horizon radius. This of course is first order surface gravity which is constant. As well, it was seen throughout that the horizon energy flux was

$$\dot{E} = \frac{r_H}{2},$$

which upon simple integration gives the expression for the energy crossing the horizon.

Summarizing the goal and achievement of this thesis is simple, and concise. The conditions of slow evolution, as have been previously established in a general way have been considered here for three distinct and physically interesting spacetimes. The application of those conditions to the Vaidya, Tolman Bondi, and Tidally Distorted spacetimes led to relevant physical implications for their slow evolution. From that, followed the physical zeroth and first laws for the respective black hole spacetimes. These were of the form predicted for the slowly evolving horizon.

5.3 Philosophy and Applications

On a philosophical level, there comes a certain degree of comfort in the notion of a slowly evolving horizon and it's classification discussed here. As already repeated, the slowly evolving regime is analogous to quasi-equilibrium thermodynamics. A simple theme that underlies these two scenarios is that in physics, one can only expect to detect the properties of an object or system if they are not changing. More importantly, if the rate of change of the system is relatively small, or slowly evolving, these properties can be reasonably approximated. Whether it be a black hole or a beaker of water, attempting to change the system drastically either by adding infalling mass to the black hole, or putting the beaker of water on a Bunsen burner, one cannot determine its physical characteristics unless in begins to settle down near equilibrium.

An opportunity for an application for slowly evolving horizons, which has not yet been discussed in the literature, is the detection of and experimental evidence for black holes and the laws of black hole thermodynamics. The theory of black holes says that they are thermodynamic objects. Experimental tests or measurements would obviously be made using the laws of slowly evolving horizons where one could measure surface gravity. The slowly evolving version of the surface gravity is one that is detectable or measurable, unlike something like the trapping gravity of [15].

More practically, the classification and laws outlined in this thesis will find application in numerical relativity, specifically when black holes reach near equilibrium states. Numerical relativity attempts to describe general relativity by getting numerical solutions to Einstein's equations. Computations in numerical relativity will likely be more efficient when the implementing the SEH laws after testing that the slowly evolving conditions are satisfied. It is probably this application that will find the most benefit in further research on slowly evolving horizons.

Analogies between black hole thermodynamics and thermodynamics run deep. SEHs take that analogy big steps further. The slowly evolving spacetimes present a regime for quasi-equilibrium black hole thermodynamics. These two regimes, black hole quasi-equilibrium and classical quasi-equilibrium, are both physical and local and there governing laws are similar as already described in Chapter 1. This is truly important because it may be true that the SEH first law is true also for non-equilibrium thermodynamics as stated Pippard [5].

One property of thermodynamics, that is useful in the lab, is that it is obvious or at least intuitive when when a given physical process is within the regime of quasi-equilibrium thermodynamics. Thus, in most cases, scientists in the lab have an intuition for when the rate of change of a physical system is comparable to the time scale of measurement. Physicists, who use thermometers and make physical measurements on a regular basis, must be working within the quasi-equilibrium regime. Most experiments occur in the quasi-equilibrium regime or at least are controlled in such a way. Similarly, the field of black hole thermodynamics can hope to have a similar understanding of black hole spacetimes and what the relative sizes of the astronomical quantities should will ensure slowly evolution. For example, becoming familiar with what the initial conditions black hole mass, dust shell density and area of a Vaidya spacetime permit slow evolution. After much study of SE spacetimes there will develop an intuition for what these astronomical quantities should be. This thesis is hopefully a beginning into such progress along. This need to build an intuition is one of the areas of further research required in the are of slowly evolving horizons.

5.4 Further Study

The study of slowly evolving horizons thus far is by no means extensive but the considerations above suggest that formalism has much promise in providing a physical description of black hole thermodynamics and identifying a spacetime or process as fitting into the regime means for concise and simple laws of black hole thermodynamics. There are certainly particular areas for which additional study could be directed.

Determining how slowly evolving horizons can generally be implemented in numerical computations will allow those in the field to more easily apply the regime. Those in the field can implement a criteria in their computations that check to see the slow evolving conditions are met and then will be able to proceed much more efficiently.

There is some investigation still required into the uniqueness of black hole horizons in general and the slicing of spacetimes in particular. This includes the trapping and slowly evolving horizons. A discussion about uniqueness of Dynamical horizons, which is relevant to the slowly evolving case, is made in [24]. As was stated by Ashtekar et al. the issue of uniqueness of these horizons is still an open one.

While Chapter 1 motivates extensively by covering thermodynamics, all the analogies and connections with it and black hole thermodynamics are not explored in this thesis. This task seems somewhat daunting since non-equilibrium thermodynamics has only relatively recently been developed and it has not been studied extensively.

More thought can go into the connection between the laws of dynamical horizon black hole thermodynamics and that of the slowly evolving and isolated cases. This discussion will obviously be an important one. It may be true that the thermodynamic laws for isolated, slowly evolving, and dynamical horizons are well established however, a discussion on how the surface gravity and angular momentum of each regime relate is required. Theoretically this could be analogous to a discussion about the connection between classical equilibrium thermodynamics to non-equilibrium thermodynamics.

Most notably though is the need to further study of the physical examples of the regime and building an intuition for the slowly evolution as described in §5.3. One such paper (in submission) by Kavanagh and Booth [36], studies this for the spacetimes covered here within. Establishing an intuition for SE analogous to thermodynamics will prove to make the SE regime of extreme practical use in real astronomical processes.

Appendix A

Spherical Harmonics and Irreducible Tidal Fields

$Y^{2,0} = -\left(3\cos^2\theta - 1\right)$	$\mathcal{E}^q_0 = \frac{1}{2}\left(\mathcal{E}_{11} + \mathcal{E}_{22}\right)$	$\mathcal{B}^q_0 = \frac{1}{2}\left(\mathcal{B}_{11} + \mathcal{B}_{22}\right)$
$Y^{2,1c} = 2\sin\theta\cos\theta\cos\phi$	$\mathcal{E}^q_{1c} = \mathcal{E}_{13}$	$\mathcal{B}^q_{1c} = \mathcal{B}_{13}$
$Y^{2,1s} = 2\sin\theta\cos\theta\sin\phi$	$\mathcal{E}^q_{1s} = \mathcal{E}_{23}$	$\mathcal{B}^q_{1s} = \mathcal{B}_{23}$
$Y^{2,2c} = \sin^2\theta\cos 2\phi$	$\mathcal{E}^q_{2c} = \frac{1}{2}\left(\mathcal{E}_{11} - \mathcal{E}_{22}\right)$	$\mathcal{B}^q_{2c} = \frac{1}{2}\left(\mathcal{B}_{11} - \mathcal{B}_{22}\right)$
$Y^{2,2s} = \sin^2\theta\sin 2\phi$	$\mathcal{E}^q_{2s} = \mathcal{E}_{12}$	$\mathcal{B}^q_{2s} = \mathcal{B}_{12}$
$Y^{3,0} = -\left(5\cos^3\theta - 3\cos\theta\right)$	$\mathcal{E}^o_0 = \frac{1}{2}\left(\mathcal{E}_{113} + \mathcal{E}_{223}\right)$	$\mathcal{B}^o_0 = \frac{2}{3}\left(\mathcal{B}_{113} + \mathcal{B}_{223}\right)$
$Y^{3,1c} = -\frac{3}{2}\sin\theta\left(5\cos^2\theta - 1\right)\cos\phi$	$\mathcal{E}^o_{1c} = \frac{1}{2}\left(\mathcal{E}_{111} + \mathcal{E}_{122}\right)$	$\mathcal{B}^o_{1c} = \frac{2}{3}\left(\mathcal{B}_{111} + \mathcal{B}_{122}\right)$
$Y^{3,1s} = -\frac{3}{2}\sin\theta\left(5\cos^2\theta - 1\right)\sin\phi$	$\mathcal{E}^o_{1s} = \frac{1}{2}\left(\mathcal{E}_{112} + \mathcal{E}_{222}\right)$	$B^o_{1s} = \frac{2}{3}\left(\mathcal{B}_{112} + \mathcal{B}_{222}\right)$
$Y^{3,2c} = 3\sin^2\theta\cos\theta\cos 2\phi$	$\mathcal{E}^o_{2c} = \frac{1}{2}\left(\mathcal{E}_{113} - \mathcal{E}_{223}\right)$	$\mathcal{B}^o_{2c} = \frac{2}{3}\left(\mathcal{B}_{113} - \mathcal{B}_{223}\right)$
$Y^{3,2s} = 3\sin^2\theta\cos\theta\sin 2\phi$	$\mathcal{E}^o_{2s} = \mathcal{E}_{123}$	$\mathcal{B}^o_{2s} = \frac{4}{3}\mathcal{B}_{123}$
$Y^{3,3c} = \sin^3\theta\cos 3\phi$	$\mathcal{E}^o_{3c} = \frac{1}{4}\left(\mathcal{E}_{111} - 3\mathcal{E}_{122}\right)$	$\mathcal{B}^o_{3c} = \frac{1}{3}\left(\mathcal{B}_{111} - 3\mathcal{B}_{122}\right)$
$Y^{3,3s} = \sin^3\theta\sin 3\phi$	$\mathcal{E}^o_{3s} = \frac{1}{4}\left(3\mathcal{E}_{112} - \mathcal{E}_{222}\right)$	$\mathcal{B}^o_{3s} = \frac{1}{3}\left(3\mathcal{B}_{112} - \mathcal{B}_{222}\right)$

Table A.1: Spherical harmonics and the harmonic components of the tidal fields as defined in terms of their fram components.

$\mathcal{E}^q = \sum_m \mathcal{E}^q_m Y^{2m}$	$\mathcal{E}^o = \sum_m \mathcal{E}^o_m Y^{3m}$
$\mathcal{E}^q_A = \frac{1}{2}\sum_m \mathcal{E}^q_m Y^{2m}_A$	$\mathcal{E}^o_A = \frac{1}{3}\sum_m \mathcal{E}^o_m Y^{3m}_A$
$\mathcal{E}^q_{AB} = \sum_m \mathcal{E}^q_m Y^{2m}_{AB}$	$\mathcal{E}^o_{AB} = \frac{1}{3}\sum_m \mathcal{E}^o_m Y^{3m}_{AB}$
$\mathcal{B}^q_A = \frac{1}{2}\sum_m \mathcal{B}^q_m X^{2m}_A$	$\mathcal{B}^o_A = \frac{1}{3}\sum_m \mathcal{B}^o_m X^{3m}_A$
$\mathcal{B}^q_{AB} = \sum_m \mathcal{B}^q_m X^{2m}_{AB}$	$\mathcal{B}^o_{AB} = \frac{1}{3}\sum_m \mathcal{B}^o_m X^{3m}_{AB}$

Table A.2: Decomposition of the quadrupole and octupole tidal fields of Table A.1.

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