Flight in a viscous fluid: Asymptotic theory of the vortex wake

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Solutions for the three-dimensional distributions of vorticity in the wake behind localized forces moving in a viscous fluid are obtained. The forces simulate the drag, thrust, and lift forces applied on the fluid by a flying animal such as a small insect. Concentrated vortex tubes in the wake are also illustrated in a visualization experiment where a "virtual" insect is modeled using the electromagnetic method of forcing. © *2005 American Institute of Physics*.
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Compact vortex structures are generated in a viscous fluid when some distribution of force is applied locally. If such a distribution can be reduced to a single force applied at a point, the generic types of vortex structures generated by the force are a vortex dipole in a two-dimensional (2D) geometry or an axisymmetric vortex ring in 3D. The solutions for the problems where the forcing can be formally presented by the Dirac δ function were obtained previously by different authors (Slezkin,[1] Landau and Lifshitz,[2] Sozou,[3] Cantwell,[4] and Voropayev and Afanasyev[5]). If the forcing translates in the fluid with constant speed generating a vortical wake, similar solutions can be obtained in the Oseen approximation.[6,7] These solutions describe the far-field wake behind a bluff body moving in fluid. Different moving objects can be effectively reduced to a combination of spatially localized forces acting on the fluid within some (moving) volume if one is interested in the flow sufficiently far from the object, at distances more than a few typical body sizes. The force moving with a velocity U is equivalent to the problem where the force is fixed at the origin while the fluid is moving with the velocity $-U$ in the opposite direction (say along the x axis). Vorticity ($\boldsymbol{\omega}=\text{curl}\,\mathbf{u}$, where $\mathbf{u}$ is the velocity vector) generated by the forcing is then transported away by the flow (advectively) and diffuses due to viscous effects. The problem is therefore to solve the equations of motion (the Navier–Stokes equations) with the appropriate point forcing in the right-hand side. This problem is by no means simple but it can be reduced to a linear problem if we neglect the transport of vorticity by the flow itself (self-induction) and consider only the transport by the uniform stream of velocity $-U$. This constitutes the Oseen approximation which is valid when the viscous effects are important and the self-induced velocity is less than U. The relative importance of advection and diffusion is characterized by the Reynolds number, $\text{Re}=UL/\nu$, where L is a typical size of the object and ν is the kinematic viscosity of the fluid.

A moving bluff body experiences a drag force. The effect of the body on the fluid is therefore described by the reaction force of the same magnitude and acting in the opposite direction. If the body is self-propelled, the reaction to the thrust force acting on the fluid in the direction opposite to that of the translation of the body should be considered as well. Previously, only the solutions for the single force which represents the drag force or the force doublet which represents the drag-thrust couple have been considered.[6] Herein we apply this approach to obtain the solution for the lift force which is applied in the direction perpendicular to that of the translation of the body. Within the general context of flows generated by localized forcing, some particular applications should be mentioned. These applications include the quantitative description of wakes behind swimming microorganisms or flying insects. Flying or swimming animals are known to create compact vortices in their wakes including vortex rings or dipoles as they apply force periodically on the fluid by flapping their wings or tails. Lighthill[8] was the first to conjecture that vortex rings should appear in animal wakes. Since then vortex structures in wakes have been observed and measured experimentally (e.g., Kokshaysky[9] and Spedding *et al.*[10]). However, details of the topology of the vortical wakes generated by the animals can be extremely complex and therefore difficult to interpret from the experimental data.

The Reynolds number is small for the motion of microorganisms ($\text{Re}\ll 1$), moderate for insects ($\text{Re}\sim 10^2$), and relatively large for birds ($\text{Re}\sim 10^4$). Our approximation is best suited for the flows with lower Reynolds numbers. For larger Re, the important difference between the solutions and the real flows is that vorticity is more concentrated, such that the vortex rings are thinner rather than being of donut-like shape as predicted by the solutions. The vortex rings also translate due to the self-induction mechanism. Surprisingly the general topology of the vorticity surfaces remains invariant and is still described well by the theoretical solution even for relatively large values of Re.

In the framework of the proposed approximation we can imagine the following physical mechanism of the flow. At each infinitesimal moment in time the force at the origin generates vorticity in the vicinity of the origin. The vorticity moves away with the fluid particles and diffuses. In the coordinate frame moving with a fluid particle the evolution of vorticity is then described simply by the solution of the diffusion equation where the vorticity was created by the force

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acting impulsively in time when the particle passed the origin of the stationary coordinate system. These solutions were obtained previously both for two and three (axisymmetric) dimensions. The last step in solving the equation of motion is to integrate the previous distributions of vorticity for all fluid particles taking into account the appropriate displacement of the particle and time delay from the moment when the vorticity was generated such that[6]

$$\boldsymbol{\omega}(x,y,z,t) = \int_0^t \boldsymbol{\omega}_{\mathbf{I}}[x - U(t-\tau), y, z, t-\tau] d\tau. \quad (1)$$

Here $\boldsymbol{\omega}_{\mathbf{I}}$ is the solution for the vorticity generated by the impulsive forcing. If the force is directed along the x axis it generates the vortex ring with y and z components of the vorticity vector[4]

$$(\omega_y, \omega_z) = \frac{I}{16\pi^{3/2}(\nu t)^{5/2}} \exp(-r^2/4\nu t)(-z,y) = \tilde{\omega}_I(-z,y), \quad (2)$$

where I is the kinematic momentum transferred to the fluid by the forcing and $r=(x^2+y^2+z^2)^{1/2}$. Similarly for the forcing directed along the z axis,

$$(\omega_x, \omega_y) = \tilde{\omega}_I(-y,x). \quad (3)$$

A flying animal experiences a drag force and generates a thrust force to overcome it. Similarly, the lift force equilibrates the weight of the body. Newton's third law requires that the reaction forces applied to the fluid by the animal are of the same magnitude but of opposite direction. While drag remains approximately constant if the animal moves with constant speed, the lift and thrust forces are applied impulsively during the downstroke of the wings. The steady-state ($t \to \infty$) solutions for constant drag or lift forces can be easily obtained in explicit form after integration of (2) or (3) as follows:

$$(\omega_y, \omega_z) = \frac{JU^{3/2}}{8\pi^{3/2}\nu^{5/2}r^{5/2}} \exp(ax) K_{-3/2}(ar)(-z,y) = \tilde{\omega} K_{-3/2}(ar)(-z,y) \text{ (drag)}, \quad (4)$$

$$\omega_x = -y\tilde{\omega}K_{-3/2}(ar),$$

$$\omega_y = x\tilde{\omega}K_{-3/2}(ar) - r\tilde{\omega}K_{-1/2}(ar) \text{ (lift)}. \quad (5)$$

Here J is the intensity of the continuous force which delivers constant kinematic momentum flux per unit time, $a=U/2\nu$, and K is the modified Bessel function of the second kind.

If we specify the magnitude of the force to vary harmonically as $\sin(\Omega t)$, the solutions can be obtained in a similar manner by integration of impulsive solutions whose magnitude is modulated by the same harmonic factor. For the forces directed along the x or z axis we obtain

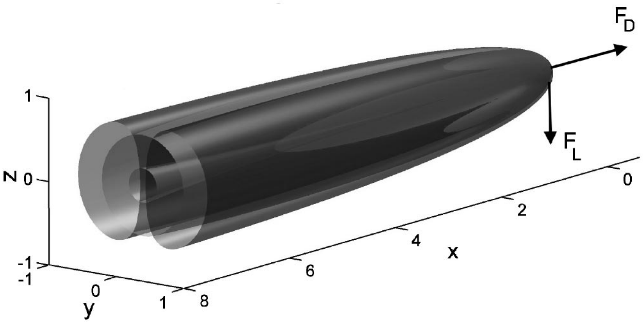

FIG. 1. Rear view of the vorticity isosurfaces generated by constant drag and lift forces applied at the origin of the coordinate system.

$$(\omega_y, \omega_z) = \tilde{\omega} \,\mathrm{Im}[b^{3/2}K_{-3/2}(abr)](-z,y) \text{ (thrust)}, \quad (6)$$

$$\omega_x = -y\tilde{\omega}\,\mathrm{Im}[b^{3/2}K_{-3/2}(abr)],$$

$$\omega_y = x\tilde{\omega}\,\mathrm{Im}[b^{3/2}K_{-3/2}(abr)] - r\tilde{\omega}\,\mathrm{Im}[b^{1/2}K_{-1/2}(abr)] \text{ (lift)}, \quad (7)$$

where $b=(1-4i\nu\Omega/U^2)^{1/2}$. Arbitrary time dependence of the force magnitude can then be taken into account using the sum of Fourier harmonics (4)–(7).

To illustrate the topology of vorticity in the wake consider specific values of control parameters which could characterize the flight of a small insect. If an insect of size $L=0.5$ cm moves with the velocity $U=15$ cm/s through the air, it experiences a drag force $F_D=C_D L^2 \rho U^2/2 = 5\times 10^{-2}$ dynes. Here we estimated the drag coefficient C_D to be approximately 1.5 as that for a sphere for Re=50. Suppose that the weight of the body is of the same magnitude. The average thrust and lift forces will then be equal to the drag force as well. Typical vortices generated by constant drag and lift forces are illustrated in Fig. 1. The vortex due to the drag force is a cone with a hole inside and is in fact a vortex ring stretched along the x axis by the "wind." The diameter of the cone increases downstream due to the viscous diffusion of vorticity. The vortex due to the lift force is a horseshoe-like vortex. The cross section of this horseshoe is a dipole which will propagate downwards by the mechanism of self-induction if it is allowed to do so. This vortex is of the same dynamical origin as the tip vortices of an airplane (low aspect ratio wing) or the vortices observed in smoke rising from a chimney in a crosswind. If the insect applies the lift and thrust forces only in a certain time interval during the wing flapping cycle (say during the downstroke), distinct vortex rings are formed in the wake [Fig. 2(a)]. The rings formed during the downstrokes, when lift and thrust prevail over the drag, are oriented at 45° to the horizontal plane (the magnitudes of lift and drag forces are equal in this case). The rings in between these are mostly due to drag and are aligned with the x axis. Insects and birds (perhaps to a larger degree) can also perform gliding flight just like airplanes do. In this case between the strokes the wings are not moving but the angle of attack is chosen so as to provide just enough lift to support the weight and to not

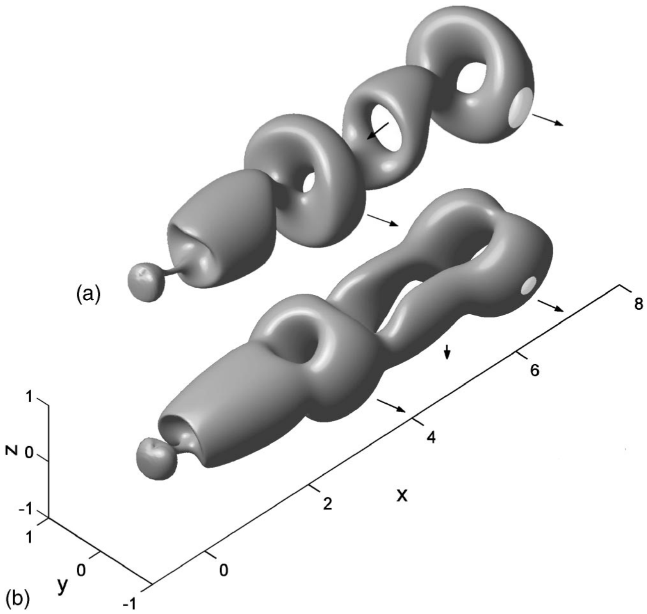

FIG. 2. Three-dimensional surface of constant vorticity $\omega^2 = 1\ s^{-2}$. (a) Front view of the isosurface of the total vorticity squared when the lift and thrust forces are applied in pulses while the drag force remains constant. (b) Same as in (a) but the lift force remains constant between the pulses. The direction of the momentum transfer by the vortex rings is indicated by arrows. Scale is in cm. $F_D = \rho J = 0.05$ dynes, $U = 15$ cm/s, $\nu = 0.15\ cm^2/s$, the average thrust and lift forces are equal to the drag force, square pulses are of duration of 1/5 of the period of the main cycle $T = 2\pi/\Omega = 0.07$ s.

lose altitude. This scenario is modeled in Fig. 2(b) where two vortex tubes of oppositely signed x component of vorticity are formed between the vortex rings.

Laboratory experiments can elucidate the effects of self-induction, tilting, and reconnection of vortex tubes as well as their interactions with a free surface that are not accounted for by the theoretical solution. Localized forcing can be modeled in the laboratory using an electromagnetic method. If an electric current flows through saline water and a magnetic field (such as from a permanent magnet) is applied locally to the fluid, the resulting force on the fluid will be in the direction perpendicular to both the current and the magnetic field. By varying the magnitude of the current and the orientation of the magnetic field we can simulate different force configurations as well as time dependences. Here we demonstrate the results of qualitative visualization experiments where a bluff body of relatively small size (diameter 0.7 cm) moves with constant speed in water. The body applies constant drag force on the fluid. A small rare-earth permanent magnet in the rear end of the body provides a magnetic field in the streamwise direction while the electric current flows in a spanwise direction between two electrodes in a tank. The resulting (Lorentz) force on the fluid is then directed downwards. It generates two vortex tubes of opposite sign [Fig. 3(a)]. A starting vortex ring can also be seen at the extreme right in Fig. 3(a). If the lift force is applied periodically during some time interval to simulate the downstrokes of the wings of the insect, the interconnected vortices in the form of the Greek letter Ω are formed [Fig. 3(b)]. A sketch in Fig. 4 illustrates the topology of these Ω vortices. This experiment clearly demonstrates the effect of self-

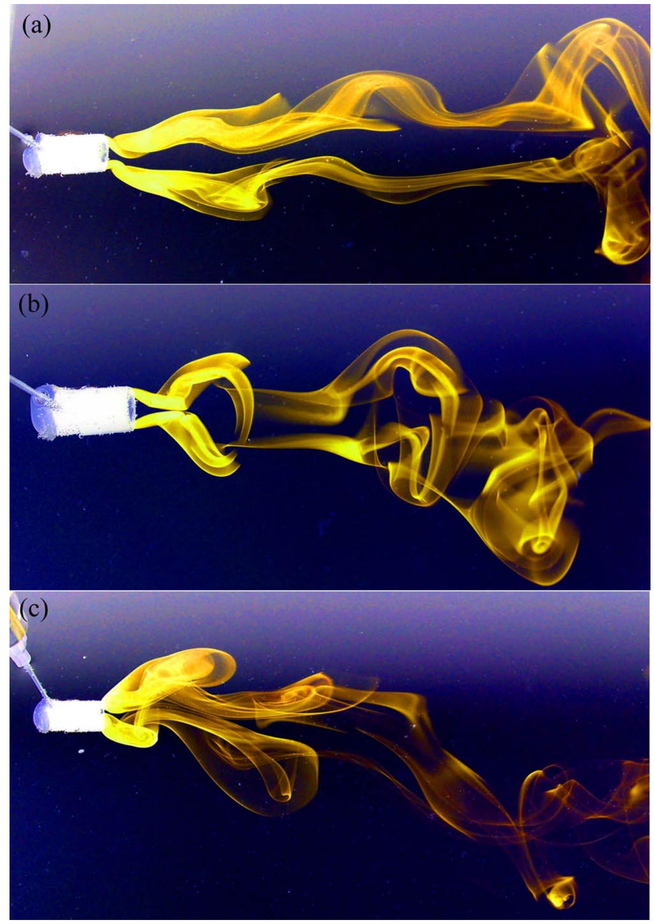

FIG. 3. (Color online). Vortex tubes in the wake of a body moving in water and with the lift force applied vertically downwards. Top view. (a) Lift force is applied continuously, velocity of the body is $U = 0.5$ cm/s. (b) Lift force is applied periodically during the time intervals of 3 s with the same interval between the forcing, $U = 0.5$ cm/s. (c) Same as in (b) but for larger velocity $U = 1$ cm/s.

induction of vortices. The upper arc of the Ω is advected downwards such that the entire structure appears to be inclined with respect to the horizontal plane. The theoretical solution which does not take into account this effect predicts the Ω vortices lying strictly in the horizontal plane (if the thrust force is not considered). For the flows in Figs. 3(a) and 3(b) the Reynolds number (Re=30) is below the threshold value when instability in the form of periodic vortex shedding starts. For larger values of the Reynolds number (Re =60) vortex tubes generated by the lift force become recon-

FIG. 4. Sketch of the vortex tubes behind a body applying the lift force periodically.

nected with the vortices which occur due to the instability [Fig. 3(c)]. The flow in this regime is topologically more complex. The dynamics of the flow where vorticity generated by periodic shedding due to instability interacts with the vorticity generated by periodic lift (and thrust) forces is not yet completely understood. This problem is important due to its possible practical applications and is a subject of further investigation.

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