

Wakes behind towed and self-propelled bodies: Asymptotic theory

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Solutions for the (steady or unsteady) wake flows induced by a localized single force or a force doublet in a uniform stream are obtained in Oseen approximation for two-dimensional (planar) as well as for three-dimensional (axisymmetric) flows. These solutions are compared with the steady solutions obtained previously by other authors in boundary layer approximation. The straightforward approach to the general problem of the flow induced by any distribution of localized forces which was developed here, can be used to obtain the vorticity and stream function distributions for the flows generated by forcing of more complicated spatial symmetry in both two and three dimensions. © 2004 American Institute of Physics. [DOI: 10.1063/1.1768071]

The problem of quantitative description of characteristics of vortical wakes arises in a large number of applications. Examples include wakes behind swimming microorganisms or flows of larger scale generated by towed or self-propelled bodies. The general feature of these flows is that different moving objects can be effectively reduced to a combination of spatially localized forces acting on the fluid if one is interested in the flow sufficiently far from the object (far-field flow). A towed object experiences a drag force. The effect of the object on the fluid is therefore described by the force equal to the drag force in magnitude and acting in the opposite direction. The wake of the towed body is characterized by a nonzero momentum. For self-propelled bodies moving with constant speed, thrust must be equal to the drag force. Two forces of equal magnitude acting in opposite directions and separated by some distance constitute a force doublet. The net momentum of the wake induced by a force doublet is zero. If particular details of the distribution of forcing inside the volume where the forces are applied are of no relevance, the forcing can be formally presented by the Dirac delta function $\delta(\mathbf{x})$. The problem of the flows induced by nontranslating single force or a force doublet applied in a point was considered by different authors (Slezkin,¹ Landau and Lifshitz,² Sozou,³ Cantwell,⁴ and Voropayev and Afanasyev⁵). It was demonstrated that a single force generates a vortex dipole in two-dimensional geometry or an axisymmetric vortex ring in three dimensions. If the force acts continuously the jet flow (starting jet) develops behind the propagating vortex front (dipole or vortex ring). A force doublet induces a flow of spatial symmetry of higher order, namely, a vortex quadrupole⁶ [in two dimensions (2D)] or two colliding vortex rings⁷ (in 3D). Herein we use the appropriate solutions obtained previously for such flows in order to obtain the solutions for the case when the forcing translates in fluid with a constant speed generating a vortical wake. The (stationary) solutions for this problem have been recently obtained by Smirnov and Voropayev⁸ in a boundary

layer approximation. In this paper we report on the results of a more general approach to this problem which allows us, in particular, to consider nonstationary flows (objects suddenly starting the motion). This approach does not require the boundary layer approximation and as a result not only longitudinal velocity but also the transverse velocity in the wake is obtained. Vorticity ($\boldsymbol{\omega} = \text{curl} \mathbf{u}$) and the stream function ($u_x = \partial\psi/\partial y$, $u_y = -\partial\psi/\partial x$) for the vortex dipole (2D) induced by the impulsive single force which acts for a short period of time delivering finite kinematic momentum I ($[I] = L^3 T^{-1}$) can be obtained in Stokes approximation,⁴

$$\omega = \frac{Iy}{8\pi(\nu t)^2} e^{-\xi^2}, \quad (1)$$

$$\psi = \frac{Iy}{2\pi(x^2 + y^2)} (1 - e^{-\xi^2}). \quad (2)$$

Here the force is located in the origin of the Cartesian coordinate system (x, y) and directed along the x axis, ν is the kinematic viscosity and $\xi = \sqrt{(x^2 + y^2)}/4\nu t$. Since (1) and (2) represent the solutions of the linearized equations of motion, any superposition of these solutions also satisfies the equations. The vorticity field for the flow induced by two forces acting in opposite directions and separated by distance ε is then given by

$$\omega_2(x, y, t) = \omega(x, y, t) - \omega(x + \varepsilon, y, t). \quad (3)$$

In the limit $\varepsilon \rightarrow 0$ such that the intensity of the force doublet is given by $M = \lim_{\varepsilon \rightarrow 0, I \rightarrow \infty} I\varepsilon$, (3) renders the solution for the impulsive vortex quadrupole,

$$\omega_M = -\frac{Mxy}{32\pi(\nu t)^3} e^{-\xi^2}. \quad (4)$$

By the same limiting procedure, the stream function is obtained in the form

$$\psi_M = -\frac{Mxy}{\pi(x^2 + y^2)^2} [1 - (1 + \xi^2)e^{-\xi^2}]. \quad (5)$$

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Alternatively, this solution can be obtained by solving the equations of motion under the assumption of the quadrupolar spatial structure of the solution (e.g., Voropayev *et al.*⁶).

Vorticity and stream function for the impulsive mushroom-like vortex (vortex ring) (3D) were also obtained in Stokes approximation,⁴

$$\omega_\varphi = \frac{I r}{16 \pi^{3/2} (\nu t)^{5/2}} e^{-\xi^2}, \tag{6}$$

$$\psi_\varphi = \frac{I r^2}{2 \pi^{3/2} (x^2 + r^2)^{3/2}} \left(\frac{\pi^{1/2}}{2} \operatorname{erf}(\xi) - \xi e^{-\xi^2} \right). \tag{7}$$

The solution is axisymmetric and does not depend on the azimuthal coordinate φ in the spherical polar coordinate system (R, θ, φ) . Here $R^2 = x^2 + r^2$ where r is the distance from the x axis and $\xi = \sqrt{(x^2 + r^2)/4\nu t}$. The dimensions of the intensity of the source I ($[I] = L^4 T^{-1}$) are different from those in 2D case. A stream function ψ_φ is introduced in standard form such that the components of velocity in spherical coordinate system are given by

$$u_R = \frac{1}{R^2 \sin \theta} \frac{\partial \psi_\varphi}{\partial \theta}, \quad u_\theta = \frac{1}{R \sin \theta} \frac{\partial \psi_\varphi}{\partial R}. \tag{8}$$

The solution for the axisymmetric flow induced by the force doublet can again be obtained by the limiting procedure in the form

$$\omega_\varphi = \frac{M x r}{32 \pi^{3/2} (\nu t)^{7/2}} e^{-\xi^2}, \tag{9}$$

$$\psi_\varphi = \frac{3 M x r^2}{2 \pi^{3/2} (x^2 + r^2)^{5/2}} \left[-\frac{\pi^{1/2}}{2} \operatorname{erf}(\xi) + \xi e^{-\xi^2} \left(1 + \frac{2}{3} \xi^2 \right) \right]. \tag{10}$$

The next step in our analysis is to obtain the solutions for the single force or force doublet moving with constant velocity U along the x axis. The solutions for (1) and (6) for vorticity in the flow induced by a stationary force were obtained by solving a diffusion equation for vorticity. For the moving source or, equivalently, for the stationary source in the spatially uniform stream of velocity $-U$, the diffusion-advection equation

$$\frac{\partial \omega}{\partial t} + U \frac{\partial \omega}{\partial x} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{11}$$

has to be solved. This equation corresponds, in fact, to the Oseen approximation (e.g., Batchelor⁹). It is straightforward to solve Eq. (11) when the solutions for the impulsive (stationary) force are known. For a forcing source in the stream, the vorticity distribution at any time can be found by integration of previous distributions due to the linearity of the solutions (see, e.g., Farlow¹⁰). The solution for the 2D force doublet, for example, is given by

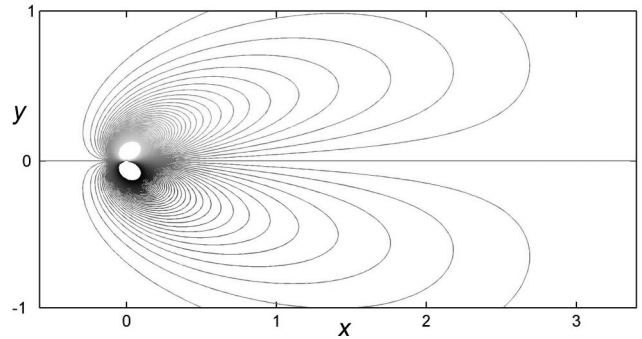


FIG. 1. Vorticity field of the 2D flow generated by a single force in a uniform stream for $J=0.01 \text{ cm}^3/\text{s}^2$, $U=0.1 \text{ cm/s}$, $\nu=0.01 \text{ cm}^2 \text{ s}^{-1}$, $t=25 \text{ s}$. Contours are from -1 to 1 with the interval 0.02 s^{-1} . Scale in cm.

$$\omega_Q(x, y, t) = -\frac{Q y}{32 \pi \nu^3} \int_0^t \frac{x - U(t - \tau)}{(t - \tau)^3} \times \exp \left[-\frac{[x - U(t - \tau)]^2 + y^2}{4 \nu(t - \tau)} \right] d\tau. \tag{12}$$

Here Q ($[Q] = L^4 T^{-2}$) is the intensity of the force doublet acting continuously in time. The solutions for single force or force doublet, both in 2D and 3D, are obtained in a similar manner by integrating the corresponding expressions (1)–(7), (9) for vorticity or the stream function such that

$$\begin{bmatrix} \omega_{J,Q} \\ \psi_{J,Q} \end{bmatrix} (x, y, t) = \int_0^t \begin{bmatrix} \omega_{I,M} \\ \psi_{I,M} \end{bmatrix} [x - U(t - \tau), y, t - \tau] d\tau. \tag{13}$$

Note that the intensity of forcing changes from the impulsive I (or M) to continuous J (or Q). Here J is the intensity of the continuous single force which starts at $t=0$ and then delivers kinematic momentum flux (force per unit mass and unit depth) equal to $J = \text{const}$. The dimensions of J are $[J] = L^3 T^{-2}$ (2D) or $[J] = L^4 T^{-2}$ (3D). Neglecting the “end effects” (the startup of the flow), we can obtain steady-state solution letting $t \rightarrow \infty$. The explicit form of the solution for the single force in 2D is given by

$$\omega(x, y, t) = -\frac{J U}{4 \pi \nu^2} \frac{y}{\sqrt{x^2 + y^2}} \exp \left(\frac{x U}{2 \nu} \right) K_1 \left(\frac{U}{2 \nu} \sqrt{x^2 + y^2} \right), \tag{14}$$

where $K_1(z)$ is a modified Hankel function.

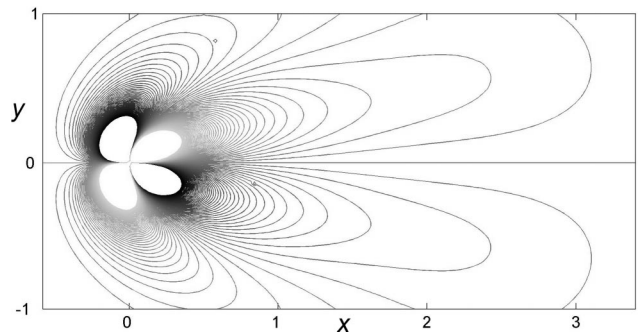


FIG. 2. Vorticity field of the 2D flow generated by a force doublet in a uniform stream for $Q=0.001 \text{ cm}^4/\text{s}^2$, $U=0.1 \text{ cm/s}$, $\nu=0.01 \text{ cm}^2 \text{ s}^{-1}$, $t=25 \text{ s}$. Contours are from -5 to 5 with the interval 0.1 s^{-1} . Scale in cm.

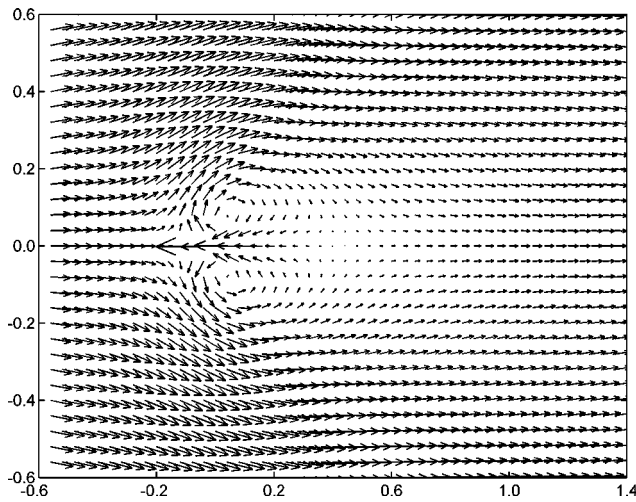


FIG. 3. Total velocity field of the 2D flow generated by a single force in a uniform stream for $J=0.01 \text{ cm}^3/\text{s}^2$, $U=0.1 \text{ cm/s}$, $\nu=0.01 \text{ cm}^2 \text{ s}^{-1}$, $t=25 \text{ s}$. Scale in cm.

We now consider the 2D solutions for single force and force doublet in more detail in order to demonstrate typical features of these flows. Vorticity fields for these two cases (Fig. 1 and 2) show a dipolar (quadrupolar) pattern near the origin where the forcing is located. The wake establishes with time behind the sources. The integration was performed to a time large enough for the end effects to be out of the domain and the flow to be considered approximately steady. The total velocity fields (u_x+U, u_y) consisting of the sum of the (perturbation) velocity (u_x, u_y) which is obtained by the differentiation of the corresponding stream function with respect to y or x and the velocity of the stream U are presented in Figs. 3 and 4. It is interesting to compare the velocity profiles in the wake region of the flows with the previous results by other authors obtained in a boundary layer approximation. The well-known solution for the longitudinal velocity perturbation in the wake behind a cylinder¹¹ is

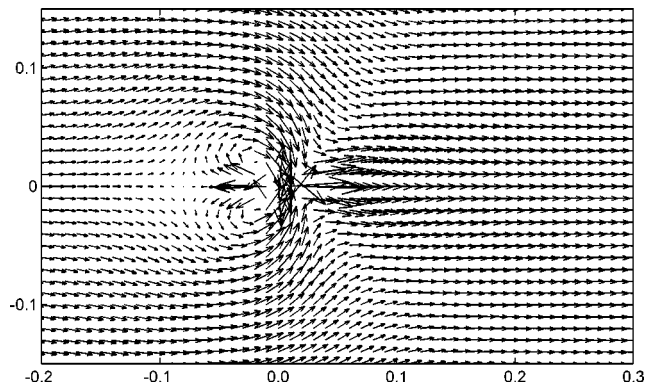


FIG. 4. Total velocity field of the 2D flow generated by a force doublet in a uniform stream for $Q=0.001 \text{ cm}^4/\text{s}^2$, $U=0.1 \text{ cm/s}$, $\nu=0.01 \text{ cm}^2 \text{ s}^{-1}$, $t=25 \text{ s}$. Scale in cm.

$$u_x = \frac{J}{2(\pi\nu U)^{1/2} x^{1/2}} e^{-\xi^2}. \tag{15}$$

The comparison (Fig. 5) of the velocity profiles given by (15) and those obtained by integration of (2) shows that there are differences both in the magnitude and the direction of the velocity. Note that a positive direction of perturbation velocity is assumed here to be the same as that of the force (in the negative direction of x axis). The velocity (15) is everywhere positive while our solution gives negative flow at the periphery of the wake. This difference may be important for studying the stability of the wake. For larger velocity of the stream the solutions [Fig. 4(b)] are closer to each other as one might expect because the boundary layer approximation becomes more justified with larger U .

The solution for the force doublet obtained in the boundary layer approximation by Smirnov and Voropayev⁸ in the form

$$u = \frac{Q}{4(\pi\nu U)^{1/2} x^{3/2}} (2\xi^2 - 1) e^{-\xi^2} \tag{16}$$

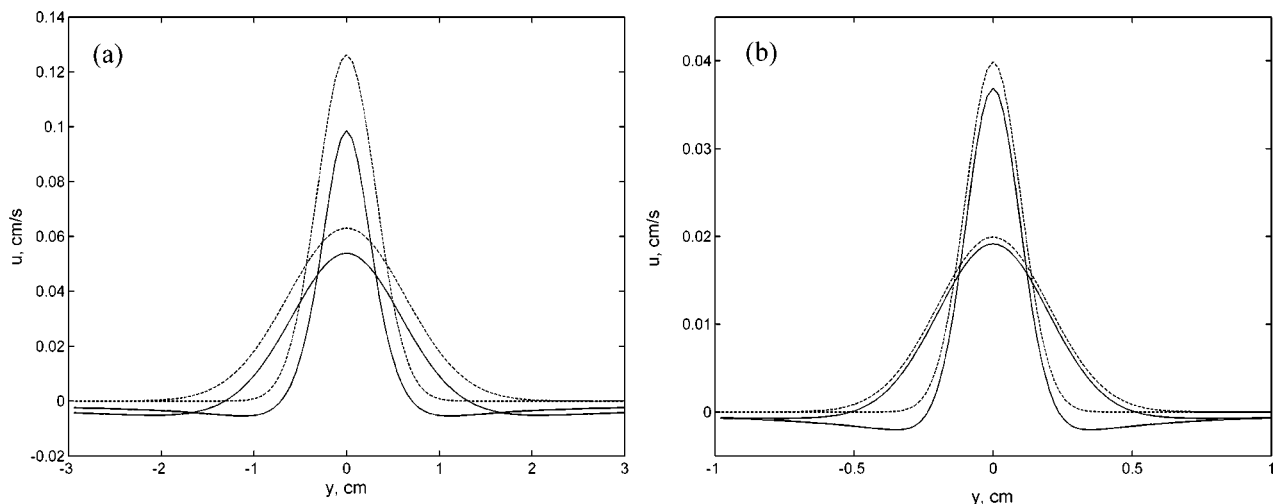


FIG. 5. Profiles of longitudinal perturbation velocity in the wake of a single force in a uniform stream for two cross sections at $x=0.5$ and 2 cm . Dashed line represents the solution (15) in boundary layer approximation while solid line represents the solution obtained by integration of (2). $J=0.01 \text{ cm}^3/\text{s}^2$, $\nu=0.01 \text{ cm}^2 \text{ s}^{-1}$, $t=25 \text{ s}$. (a) $U=0.1 \text{ cm/s}$, (b) $U=1 \text{ cm/s}$.

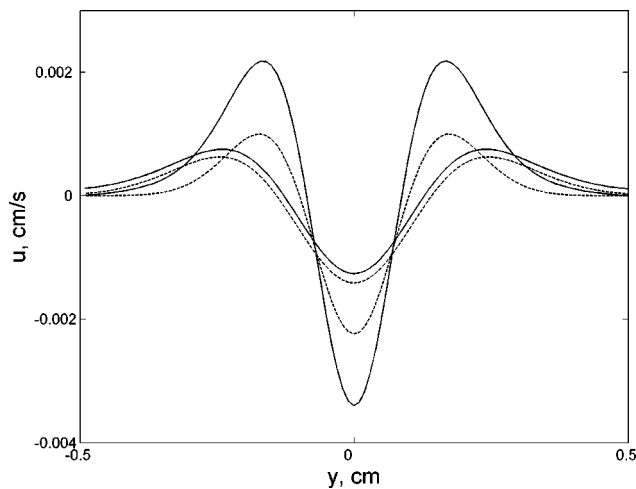


FIG. 6. Profiles of longitudinal perturbation velocity in the wake of a force doublet in a uniform stream for two cross sections at $x=0.5$ and 1 cm. Dashed line represents (16) while solid line represents the solution obtained by integration of (5). $Q=0.001$ cm³/s², $\nu=0.01$ cm² s⁻¹, $t=25$ s, $U=1$ cm/s.

is also compared (Fig. 6) with the appropriate solution obtained by integration of (5). This comparison exhibits similar features as those for a single force.

In conclusion, we have derived the solutions for the wake flows induced by a localized single force or a force doublet in a uniform stream in Oseen approximation for two-dimensional (planar) as well as for three-dimensional (axi-

symmetric) flows. This approach can be applied readily to other geometries of localized forcing. However, application of the solutions presented here to the flows with higher Reynolds number is restricted due to instability of the wakes in the form of the von Karman vortex streets.

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¹N. A. Slezkin, "On the case when the equations of motion for viscous fluid can be integrated," *Uchen. Zap. Mos. Gos. Univ.* **2**, 89 (1934) (Sci. Papers Moscow State Univ., in Russian).

²L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).

³C. Sozou, "Development of the flow field of a point force in an infinite fluid," *J. Fluid Mech.* **91**, 541 (1979).

⁴B. J. Cantwell, "Viscous starting jets," *J. Fluid Mech.* **173**, 159 (1986).

⁵S. I. Voropayev and Y. D. Afanasyev, *Vortex Structures in a Stratified Fluid* (Chapman and Hall, London, 1994).

⁶S. I. Voropayev, Y. D. Afanasyev, and I. A. Filippov, "Horizontal jets and vortex dipoles in a stratified fluid," *J. Fluid Mech.* **227**, 543 (1991).

⁷S. I. Voropayev, Y. D. Afanasyev, V. N. Korabel, and I. A. Filippov, "On the frontal collision of two round jets in water," *Phys. Fluids* **15**, 3429 (2003).

⁸S. A. Smirnov and S. I. Voropayev, "On the asymptotic theory of momentum/zero-momentum wakes," *Phys. Lett. A* **307**, 2 (2003).

⁹G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).

¹⁰S. J. Farlow, *Partial Differential Equations for Scientists & Engineers* (Dover, New York, 1993).

¹¹G. Birkhoff and E. H. Zarantonello, *Jets, Wakes and Cavities* (Academic, San Diego, 1957).