Electric Force and Potential Energy

For a charge $q_0$ in an electric field:

**The force picture**

$$\vec{F} = q_0 \vec{E}$$

Can we similarly look for an energy picture?

**The energy picture (the work-kinetic energy theorem)**

$$W_{\text{net}} = q_0 \int_i^f \vec{E} \cdot d\vec{s} \quad \text{and} \quad K_f - K_i = W_{\text{net}}$$

$$\Delta U = U_f - U_i = -W_{\text{net}} = -q_0 \int_i^f \vec{E} \cdot d\vec{s}$$
We will do a familiar problem again – projectile motion in the presence of a gravitational force – in two ways.

**The force picture**

\[ \vec{F} = m \vec{g} = -mg \hat{k} \]

**The energy picture (the work-kinetic energy theorem)**

\[ W_{\text{net}} = \int_{i}^{f} \vec{F} \cdot d\vec{s} \]

is work done by the field on the mass. Positive work reduces the potential energy.

\[ \Delta U = U_f - U_i = -W_{\text{net}} \]

Positive work increases the kinetic energy.

\[ K_f - K_i = W_{\text{net}} \]
Gravitational Force and Potential Energy

For a mass $m$ in a gravitational field:

\[ \vec{F} = m \vec{g} = -mg \hat{k} \]

The force picture

Can we similarly look for an energy picture?

The energy picture (the work-kinetic energy theorem)

\[ W_{\text{net}} = -mg \int_{i}^{f} dz = -mg (z_f - z_i) \]

\[ \Delta U = U_f - U_i = -W_{\text{net}} = mg (z_f - z_i) \]

\[ \Delta U = mg \Delta z \]
Contour Maps

are lines of **constant height** and **constant gravitational potential energy**.
Electric Potential Difference

Let's say we move a test charge from point A to point B.

\[ \Delta V = - \int_A^B \vec{E} \cdot d\vec{s} \]

For a point charge:

\[ \vec{E} = k_e \frac{q}{r^2} \]

\[ \Delta V = - \int_A^B \vec{E} \cdot d\vec{s} = -k_e q \int_A^B \frac{1}{r^2} \hat{r} \cdot d\vec{s} \]

The path denoted by the vector \( d\vec{s} \) does not have to be along the field. It is any path between the two paths A and B. It does not matter which path you choose.

In general the above integral can be complicated.
\[ \vec{F} = q \vec{E} \]

*Force has units of $N$ (newtons)*

*Charge has units of $C$ (Coulombs)*

So electric field is $N/C$

\[ \Delta V = -\int_{A}^{B} \vec{E} \cdot d\vec{s} \]

*Potential difference is*

(electric field) x (distance)

(N/C).m which is J/C (joules per coulomb)

This is also denoted as V (volts)

Therefore, units of electric field can also be written as V/m.

*Electric field: both N/C and V/m is correct.*

*Electric potential: both J/C and V is correct.*
The change in potential *energy* is proportional to the change in electric potential

\[ \Delta U_E = q \Delta V \]
Equipotentials are Voltage Contours

Uniform field between Two Charged Parallel Plates

Constant Electric Field and Equipotential Lines
Equipotentials are Voltage Contours
The Electric Field is the Electric Potential Gradient

\[ \Delta V = -\int_A^B \vec{E} \cdot d\vec{s} \]

Given the electric field: we can figure out the potential difference.

Now the inverse problem:
Given the electric potential: how do we figure out the electric field?

\[ E = \frac{k_e q}{r^2} \]

\[ V = k_e \frac{q}{r} \]

\[ \vec{E}(r) = -\frac{\partial V}{\partial r} \hat{r} \]
The Electric Field is the Electric Potential Gradient

\[ \vec{E}(r) = -\frac{\partial V}{\partial r} \hat{r} \]

Why the partial derivative?

\[ \vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k} \]

\[ = \vec{E}(x, y, z) \]

in Cartesian coordinates, or

\[ \vec{E} = \vec{E}(r, \theta, \phi) \]

in spherical coordinates

In general:

\[ \vec{E}(x, y, z) = -\nabla V = - \left[ \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z} \right] \]
Electric Potential on the Surface of a Charged Conductor

**Zero Electric Field: Inside** an electrical conductor, in electrostatic equilibrium, there is no net electric field. All the charge resides on the surface. The field lines point normal to the surface at every point on the surface.

**Equipotential: On the surface** of an electrical conductor, in electrostatic equilibrium, there is a constant electrical potential.

\[ \Delta V = -\int_A^B \vec{E} \cdot d\vec{s} \]

- \( d\vec{s} \) is tangential (parallel) to the surface
- \( \vec{E} \) is perpendicular to the surface

\[ \Delta V = -\int_A^B \vec{E} \cdot d\vec{s} = 0 \]
Conductors and Gauss' law

Zero Electric Field: Inside an electrical conductor, in electrostatic equilibrium, there is no net electric field. All the charge resides on the surface.

Infinite non-conducting charged sheet

$\bullet$

P

+ + +

Q

+ + +

R

S

What is the field at points P, Q, R and S?

Infinite conducting uncharged sheet
The van de Graaff Electrostatic Generator

Credit: Edward Hayden, Feb 17, 2010.

A willing student holds on to the metal sphere. The charge makes his hair repel and stand up.
Conductors: Electric Potential and Charge

A metal sphere (radius \( r \)) has an electrical potential that is equal everywhere on its surface.

How to relate electric field, electric potential and electric charge?

1. Closed surface: begin with Gauss' Law
Imagine drawing a Gaussian surface just outside the sphere

2. From Gauss' Law the flux is:
\[ \Phi_E = \frac{Q}{\epsilon_0} = E \left( 4\pi R^2 \right) \]

3. The electric field is:
\[ E = \frac{1}{4\pi \epsilon_0} \frac{Q}{R^2} \]

4. The electric potential is:
\[ V = \int_{\infty}^{R} \vec{E} \cdot d\vec{S} = \frac{1}{4\pi \epsilon_0} \frac{Q}{R} \]
A big metal sphere and a small metal sphere in electrical contact (e.g. with a wire joining them) will be at the same electrical potential $V$.

For the big sphere:
$$V = \frac{1}{4 \pi \varepsilon_0} \frac{Q_1}{r_1}$$

For the small sphere:
$$V = \frac{1}{4 \pi \varepsilon_0} \frac{Q_2}{r_2}$$

The big sphere has more charge than the small one.

$$\frac{Q_1}{r_1} = \frac{Q_2}{r_2}$$
A big metal sphere and a small metal sphere in electrical contact (e.g. with a wire joining them) will be at the same electrical potential $V$.

For the big sphere:

$$ V = \frac{1}{4 \pi \varepsilon_0} \frac{Q_1}{r_1} $$

$$ E_1 = \frac{1}{4 \pi \varepsilon_0} \frac{Q_1}{r_1^2} = \frac{V}{r_1} $$

For the small sphere:

$$ V = \frac{1}{4 \pi \varepsilon_0} \frac{Q_2}{r_2} $$

$$ E_2 = \frac{V}{r_2} $$

The field on the big sphere is smaller! The smaller the radius of curvature the higher the field. Hence, lightning rods are made sharp.
Why the spark?

Air is normally a very good electric insulator. But when the electric field is high, this induces breakdown (called “dielectric breakdown”) of the air. The “fresh air” smell during a thunderstorm comes from ozone after dielectric breakdown.

The left sphere is the “lightning rod”:
the smaller it is, the easier it is to generate sparks.