SUGGESTED REFINEMENTS TO COURSES ON DERIVATIVES: PRESENTATION OF VALUATION EQUATIONS, PAY OFF DIAGRAMS AND MANAGERIAL APPLICATION FOR SECOND GENERATION OPTIONS

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Abstract

This pedagogical paper should help enrich the derivatives course that are delivered to both graduate and undergraduate students. We provide pricing formulae and payoff diagrams for some of the more popular second generation or exotic options. In spite of their growing popularity, current textbooks usually provide only cursory definitions, often without supporting mathematical equations, payoff diagrams and, most importantly, examples of their managerial application. This paper presents valuation equations, payoff diagrams and managerial applications for binary, lookback, chooser, compound, Bermuda, Asian, barrier and forward start options.

Keywords: Exotic options; Managerial Applications; Valuation Equations

JEL classification: A22; C33; C39; G11

Introduction

In the last few decades exotic or second generation options have proliferated in the over-the-counter marketplace. These non-standardized products are engineered to the needs of a niche market of corporate treasurers and investment bankers. While these instruments represent a small portion of the options market, they are nonetheless useful tools in the management of a variety of corporate risks.

Textbooks on derivatives give first generation or plain vanilla options in-depth treatment, with discussion of pricing models, their derivation, and payoff or profit diagrams with appropriate boundary conditions. Typically, there is much less coverage of exotic options. Most authors of derivatives or investment textbooks restrict their discussions to cursory definitions of these products (for example Bodie, et al.), without any quantitative elaboration.

A relatively early derivatives book by Daigler shows some graphical representations of these options, including three-dimensional plots. Hull provides definitions of some exotic options but little other coverage, while his more recently released Options, Futures and Other Derivatives: Fifth Edition: 2003 describes exotic options, but provides only two payoff diagrams in the entire chapter.

The authors own full responsibility for the contents of the paper.
which are the payoffs from a short position and a long position in a range forward contract. Similarly, Strong\textsuperscript{5} provides a very limited table without any equations, payoff diagrams and only a short discussion,\textsuperscript{5} restricted largely to the definitions of these products. Jarrow and Turnbull\textsuperscript{1} provide a broader coverage, including a more detailed description of exotic options and some managerial applications. Among the most recently published texts reviewed, McDonald\textsuperscript{6} presents payoff diagrams for a barrier option and also for a gap option. Dubofsky and Miller\textsuperscript{7} relegate the topic to the end of the book, providing only limited formulae with no graphical analysis, and Carter\textsuperscript{8} chooses not to address exotic options at all. The discussion that follows provides an overview of several common exotic options listed above that, in varying degree of detail, can enrich derivatives and investment courses.

**Binary (or Digital) Option**

A binary option provides one of two payoffs: zero, if the option expires out-of-the-money, or a fixed amount if the option expires in-the-money. These are also called cash-or-nothing options. If the payoff is not cash, they are referred to as asset-or-nothing options. Calls are in-the-money when the stock price \( S \) exceeds the striking price \( X \), and vice-versa for puts. If the in-the-money payoff at expiry is denoted by \( Q \), then the current value of the call option is given by

\[
C = Q e^{-r(T-t)} N(d_1), \quad d_1 = \frac{\ln(S/X) + (r-q-\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
\]  

In the above equation, \( T \) is the expiry date, \( t \) is the current time, \( \sigma \) is the volatility of the underlying stock, \( r \) is the risk-free interest rate and \( q \) is the dividend yield. \( N \) is the cumulative normal distribution function evaluated at \( d_1 \). The profit realized at expiry is given by

\[
Q(I - e^{-r(T-t)}) N(d_2), \quad d_2 = \frac{\ln(S/X) + (r-q)(T-t)}{\sigma \sqrt{T-t}}.
\]  

The payoff diagram for a binary call option is shown by the solid line in Figure 1. The broken line represents the payoff from a plain vanilla option which pays an amount equal to the asset price if \( S > X \), but zero otherwise. As an example, let \( S_0 = 50 \), \( X = 50 \), \( \sigma = 0.3 \), \( r = 0.07 \), \( T-t = 0.25 \) years, and \( q = 0 \). Using Equation 1 the value of a claim that pays $100 if \( S > X \) in three months is $51.66.

**Figure 1**

**Payoff Diagrams**

A. Binary Call Option

![Payoff Diagram A](image1)

B. Binary Put Option

![Payoff Diagram B](image2)

**Note:** For comparison, the expected payoffs from vanilla options are shown as broken lines.

Figure 1B shows the payoff diagram for a binary put option. The payoff is \( Q \) if \( S < X \), and zero otherwise. The value of the option is thus \( P = Q e^{-r(T-t)} N(-d_2) \), and the profit realized is \( Q(1 - e^{-r(T-t)}) N(d_2) \). Using the same values for \( S_0, X, \) etc., used above, the value of a claim that pays $100 is $47.50.

\[ N(\frac{z}{\sigma \sqrt{T-t}}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{z}{\sigma \sqrt{T-t}}} \exp(-\frac{t^2}{2}) dt \]
Application

Consider a pure transmission and distribution electric utility, one that has no generation assets and must acquire power for its customers via contractual arrangements with commercial power generators. There are occasional spikes in the demand for power due to unusually hot days (e.g., greater use of air conditioning) or cold winter days (use of auxiliary electric heat). These events can lead to a temporary, but dramatic, rise in the spot rate for electricity.

A binary call option is one alternative the utility might use to partially hedge this risk. If such an option were used to hedge against the risk of hot days the option striking price might be based upon cooling degree days over the period July 1 through August 31. The option might provide a fixed payout of $50,000 if the CDDs exceed 750 and nothing if CDDs are less. This is not a perfect hedge, but an inexpensive method of risk reduction than a traditional call option.

Lookback Option

A lookback option (also called a hindsight option) appears to have 20/20 hindsight as its payoff cannot be determined until the end of the option’s life. There are two versions of the lookback option. One version has a fixed striking price that is determined at the time the option is written. The payoff is the maximum difference between the optimal stock price and the fixed exercise price. The other version has a floating exercise price that, again, is based on optimal choice. The exercise price of a floating lookback put is the maximum stock price, \( S_{\text{max}} \), reached over the life of the option, while for a call option, the exercise price is the minimum price, \( S_{\text{min}} \), reached over the option’s life. This means that a floating lookback option is never out of the money at expiration, is always exercised, and as a consequence is a relatively expensive derivative.

If \( S_T \) is the stock price at expiry, the payoff is simply \( \max(S_T - S_{\text{min}}, 0) \), as illustrated in Figure 2.

![PAYOFF DIAGRAM FOR A LOOKBACK CALL OPTION](image)

Note: The payoff is \( ST - 50 \).

* A cooling degree day is defined as the average of a day’s high and low temperature minus 65 degrees Fahrenheit. For example, if a day’s high is 95 and the low is 79 this results in 22 cooling degree days: \( 0.5(95 + 79) - 65 = 22 \).
The value of a European lookback call option is given by Hull

\[ C_{tr} = S_e^{r - q - \sigma^2 / 2} (T-t) \]

\[
= \ln(S_t / S_{min}) + (r - q - \sigma^2 / 2) (T-t) \]

\[
+ \frac{\ln(S_{max} / S_{min}) + (r - q - \sigma^2 / 2) (T-t)}{\sigma \sqrt{T-t}} \]

A similar set of equations can be obtained for the price of a European lookback put option, where the payoff is \( \max[S_{max} - S_t, 0] \). In this case

\[ P_{tr} = -S_t e^{r (T-t)} N(b_2) + S_t e^{r (T-t)} \frac{\sigma^2}{2 (r-q)} N(-b_2) \]

\[-S_{min} e^{r (T-t)} [N(b_1) - \frac{\sigma^2}{2 (r-q)} e^{\delta N(-b_2)}] \]

where \( S_{max} \) is the maximum asset price reached.

As an example, consider a lookback call option due to expire in 12 months. Let the initial stock price be \( S_0 = 50 \), and the volatility of the underlying stock 25 per cent per annum. Furthermore, let the yield on the stock be 2.5 per cent, interest rate 9 per cent and the minimum value of the underlying asset be $40. Using Equation 2, the price of the option is $13.59. For a maximum stock price of $60 the price of the corresponding lookback put option is calculated from Equation 3 to be $11.08, all other parameters being the same as before.

**Application**

A hedge fund might have identified a regional bank they believe is a likely takeover candidate, but with the timing of the tender offer uncertain. Suppose that there is concurrently substantial volatility in the broad market. To speculate on the takeover the fund could simply buy shares or plain vanilla call options. However, the market volatility means that if the fund were to buy calls today there is a good chance that the calls will be cheaper sometime in the future after a market pullback. A lookback call is an alternative that removes the “opportunity cost” associated with the market volatility. The date the hedge fund establishes the position is much less important with the lookback option because its value ensures the optimal circumstances for the option holder.

**Chooser Option**

A chooser option (also known as an as-you-like-it-option) is an option whereby the owner can decide by a specified future date whether to declare the option to be call or a put. This class of option allows the investors to straddle the market with the purchase of a single security. Straddling the market with plain vanilla options would necessitate the purchase of both a call and a put, whereas with a chooser option the investor need only purchase...
A chooser option is essentially a cheaper version of a straddle. Obviously, the payoff from a chooser option at time \( t \) will depend on whether the option is a call or a put, i.e.,
\[
\max[C(S(t),X,T),P(S(t),X,T)]
\]
where \( T_c \) and \( T_p \) represent the lifetimes of the call and put options, respectively. The analysis that follows assumes that these two lifetimes are equal. In Figure 3, the investor selects a call option at time \( t \), and the payoff is simply given by the difference between the strike price and the stock price at expiry. If \( X_c = X_p \) the option is a simple chooser option, and the payoff diagram mimics a long straddle, as shown in Figure 4.

Using put-call parity it can be shown that the payoff from a simple chooser is the same as a) buying a call with underlying asset price \( S_0 \), striking price \( X \), and time to expiry \( T \), and b) buying a put with striking price \( X(T+q)^{-1} \), where \( i \) is the elapsed time since purchase when the investor chooses between a call and put option (Rubinstein, Hull,). The price of the chooser option is given by
\[
C = S_0 e^{-qt} \left[ N(d_1) - N(-d_2) \right]
- X e^{-rT} \left[ N(d_1 - \sigma \sqrt{T}) - N(-d_2 + \sigma \sqrt{T}) \right]
\]

where
\[
d_1 = \frac{\ln(S_0/X) + (r - q + \sigma^2 / 2)T}{\sigma \sqrt{T}}
\]
and
\[
d_2 = \frac{\ln(S_0/X) + (r - q + \sigma^2 / 2)T + (r-q)(T-t)}{\sigma \sqrt{t}}
\]

For example, consider a simple chooser option with underlying asset price \( S = $150 \), striking price \( X = $150 \), time to expiration \( T = 1 \) year, interest rate = 8%, dividend rate = 6% and underlying asset volatility \( \sigma = 25\% \). Using the above formulae, the equilibrium price of the chooser option value at various times \( t \) is shown in Table 1.
FIGURE 4
PAYOFF FOR A CHOOSER OPTION AS A FUNCTION OF UNDERLYING ASSET PRICE, WITH THE CALL AND PUT OPTIONS HAVING THE SAME STRIKE PRICE, X.

Table I shows that as \( t \) increases the chooser option becomes more expensive because an investor has more information to forecast the underlying asset price. For the two extreme cases, \( t = 0 \) and \( t = 1 \), the chooser value is the same as the value of the call option and the straddle, respectively. These define maximum and minimum values of the chooser option (Rubinstein”).

**Application**

A hedge fund may decide to place a merger arbitrage bet. Pending a regulatory decision a firm’s stock is likely to either rise or decline substantially, but not both. A plain vanilla straddle is a traditional strategy for such a situation, but arguably an inferior one because...
one of the two options making up the straddle is expected to become worthless. A chooser option is a single option that takes on the character the holder wishes.

**Compound Option**

A compound option is an option on an option. Black and Scholes demonstrated that equity in a leveraged firm is an option. Therefore, buying an option on a levered firm is analogous to an option on an option.

In the derivatives markets options on futures or futures options would be close to this category of security. A compound option could be a call on a call, a call on a put, put on a call or put on a put. For the sake of brevity only the call on a call is demonstrated but the intuition is easily extended to the other three securities. Because an option will always sell for less than the underlying asset, a compound option will always sell for less than the first underlying option.

**FIGURE 5**

**PAY OFF DIAGRAMS FOR A CALL ON A CALL COMPOUND OPTION**

A. FOR AN OPTION WITH STRIKE PRICE $X_1$

![Diagram of Payoff for Option with Strike Price $X_1$]

B. FOR AN OPTION WITH STRIKE PRICE $X_2$

![Diagram of Payoff for Option with Strike Price $X_2$]

Notes: $C(T_1)$ is the value of the call option at time $T_1$, while $S(T_2)$ is the value of the underlying stock at $T_2$. The option will only be exercised if the value of the second option is greater than $X_1$.

*Prior to the opening of the CBOE in 1973 a variety of option contracts traded through the Put and Call Brokers and Dealers Association. Speculators traded in calls, puts and straddles, with the latter allowing its owner to buy or sell. These early straddles are ancestors of the chooser option.
At time $T_1$, the first strike price $X_1$ is paid to buy a call option. This gives the investor the right to buy the underlying asset for the second strike price $X_2$ on the second expiry date $T_2$. To determine the payoff, consider the payoff due to a vanilla call option, $\max[S - X_1, 0]$. The payoff at time $T_1$ is $\max[S(T_1) - X_1, 0]$ with $S$ replaced by the price of the call option at $T_1$.

Hence the payoff at time $T_1$ is given by $\max[C(S(T_1), X_1, 0), \max[S - X_1, 0]]$, and the payoff at time $T_2$ is $\max[S(T_2) - X_2, \max[C - X_2, C - X_1]]$. Figure 5 shows payoff diagrams for the option at time $T_1$ and $T_2$.

Closed-form solutions for the cost of the option, and hence the net profit realized, are described by Hull and McDonald. The option will be exercised only if the value of the second option is greater than $X_1$. McDonald defines $S^*$ to be the critical stock price, above which the compound option is exercised. In other words, the $S^*$ is obtained from the value of the first option, which is $C(S^*, T_1) = X_1$. Further define
\[ a_1 = \frac{\ln(S/S^*) + (r - \sigma^2/2)T_1}{\sigma \sqrt{T_1}} \quad \text{and} \quad a_2 = a_1 - \sigma \sqrt{T_1} \]

from which the following pricing formulae for various compound options can be derived:

**Call on Call Option**:
\[ CC = S e^{-rT_1} N(a_1, \sqrt{T_1}) - X e^{-rT_2} N(a_2) \]
\[ (-a_2, d_2; \sqrt{T_1}) + x e^{-rT_2} N(-a_2) \] (6)

**Call on Put**
\[ CP = -S e^{-rT_1} N(-a_2, d_2; \sqrt{T_1}) + X e^{-rT_2} N(a_2) \]
\[ (-a_2, d_2; \sqrt{T_1}) - xe^{-rT_2} N(a_2) \] (7)

**Put on Call**
\[ PP = S e^{-rT_1} N(a_1, -d_1; \sqrt{T_1}) - X e^{-rT_2} N(-a_2) \]
\[ (a_1, -d_1; \sqrt{T_1}) - xe^{-rT_2} N(-a_2) \] (8)

Samples prices for these four compound options are shown in Table 2.

**TABLE 2**

<table>
<thead>
<tr>
<th>PRICES FOR FOUR TYPES OF COMPOUND OPTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>For the Following Data</strong></td>
</tr>
<tr>
<td>Stock price</td>
</tr>
<tr>
<td>Exercise price to buy asset</td>
</tr>
<tr>
<td>Exercise price to buy option</td>
</tr>
<tr>
<td>Volatility</td>
</tr>
<tr>
<td>Risk-free interest rate</td>
</tr>
<tr>
<td>Expiration for Option (years)</td>
</tr>
<tr>
<td>Expiration for Underlying Option (years)</td>
</tr>
<tr>
<td>Dividend yield</td>
</tr>
<tr>
<td><strong>Compound Option Prices are</strong></td>
</tr>
<tr>
<td>Call on Call</td>
</tr>
<tr>
<td>Put on Call</td>
</tr>
<tr>
<td>Call on Put</td>
</tr>
<tr>
<td>Put on Put</td>
</tr>
</tbody>
</table>

Note: Calculated from Equations 5 - 8

*Note that $d_1$ and $d_2$ are the same as used in the Black-Scholes option pricing model for a plain vanilla option i.e., $c = N(d_1) - X \exp(-rT) N(d_2)$, where the symbols have their usual meanings.

**The cumulative bivariate distribution function is defined by the double integral $NN(a,b,c) = \int \int_{x<0,y<0} \exp(-\frac{1}{2} x^2 - \frac{1}{2} y^2 + cy) dx dy$.**
Application

The James River Company once had a convertible bond issue that was convertible into preferred stock, with the preferred issue convertible into the firm's common stock. Someone who acquired the bond also held an embedded compound option. They had a call option on preferred shares that came with an option on common shares. This type of option may be cost-effective when someone wants to hedge a risk whose nature is unclear, especially in the fixed income market. Like the chooser option, it is cheaper than a plain vanilla version.

Bermuda Option

A Bermuda option is one that may be exercised only on certain predetermined dates in addition to the final maturity date. In doing so, the option strikes a balance between American exercise, which may be exercised any time prior to maturity (continuous process), and European exercise which may be exercised only on the final maturity date (discrete process). This may be the origin of the name, Bermuda being between Europe and America. The Bermuda option is, therefore, a series of European expiry dates. If there are a large number of these fixed maturity dates with small intervals in between each, then the process would approximate American exercise, otherwise the process would be closer to an additive series of European exercise options. The payoff diagram is shown in Figure 6 and is identical to that of a vanilla option, except that the payoff (given by Max[S, - X, 0]) now depends on the value of the stock at each maturity date, t₁, t₂, t₃, etc.

FIGURE 6
PAYOFFS FOR A BERMUDA OPTION

Note: The option may be exercised only at times t₁, t₂, t₃, etc. The payoff is Sₜ - X
SUGGESTED REFINEMENTS TO COURSES ON DERIVATIVES: PRESENTATION OF VALUATION EQUATIONS

The price of this option must be determined using a compound binomial model, where the number of periods, \( n \), is equal to the time to maturity divided by the exercise frequency.

The binomial pricing formula for a call option is

\[
C = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S_0 u^j d^{n-j} - X)
\]

(9)

also known as the Cox-Ross-Rubinstein option model. The binomial probability term gives the probability of obtaining \( j \) successes in \( n \) trials. Here, \( p \) is the risk neutral probability that the price of the underlying stock will increase to \( S_u \) (McDonald\(^9\)). Conversely, \( (1-p) \) is the probability that the price will decrease to \( S_d \). \( X \) is the striking price and \( R \) is the interest rate. As an example, consider a non-dividend paying stock, with \( S = $100 \) and \( X = $100 \), \( R = 0.1 \), \( \sigma = 0.2 \), with a 6-month time to expiry. Table 3 shows the call option price for different exercise frequencies.

<table>
<thead>
<tr>
<th>Interval</th>
<th>No. of periods (( n ))</th>
<th>Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>6</td>
<td>$8.048</td>
</tr>
<tr>
<td>Weekly</td>
<td>26</td>
<td>$8.22</td>
</tr>
<tr>
<td>Daily</td>
<td>180</td>
<td>$8.27</td>
</tr>
</tbody>
</table>

Note: The results indicate that the higher the exercise frequency, the closer the Bermuda option price approaches the Black-Scholes price of $8.28.

This idea of holding the option or terminating the option is analogous to abandonment value in corporate finance as abandonment in a capital budgeting project is simply an embedded option for corporate financial managers. Robichek and Van Horne\(^{12}\) proposed that the project be abandoned in the first year that its abandonment value exceeds the present value of the remaining expected cash flows associated with the continued operation of the investment. Dyl and Long\(^{13}\) point out that there may be an even greater advantage to abandonment in a period subsequent to the first instance when abandonment value exceeds the present value of continued operations. They argue that subsequent years have to be analyzed in order to maximize the present value of abandonment. Similarly, with a Bermuda exercise, the first opportunity to exercise may not be the profit-maximizing condition. Rather, a revision of probabilities and opportunity cost of capital need to be taken into account in order to determine the potential for optimal exercise.

Application

An investment bank may facilitate a product for two customers, with the underlying asset being a deliverable commodity (such as oil) or a service (ocean freight capacity). In these circumstances the delivery of the product or the provision of the service may not conveniently be available on demand. To deal with these logistical issues while also providing semi-continuous protection the option may be designed with periodic exercise dates for the convenience of the option writer.

Asian Options

Asian options are also known as average price options in that the investor does not know the payoff until the option matures. To be valuable the option need not be in-the-money on the maturity date as in the case of a European option, nor is the option's time value terminated by premature exercise as in the potential case for American exercise. Rather, the option holder will receive a payoff based on the average price over the life of the option. If the average stock price is denoted by \( \bar{S} \), then the payoff due to the call option is \( \max(\bar{S} - X, 0) \), and for the put, the payoff is \( \max(X - \bar{S}, 0) \). These are shown graphically in Figure 7.
Closed form solutions for the price, and hence the profit realized, of Asian options assume that \( \bar{S} \) is a geometric average of the daily prices. The following equations can be used to calculate the price of European continuous geometric average options with strike price fixed (Zhang, 1):

\[
\text{Call (Asian)} = S_0 e^{-r(T-t)} \sigma \sqrt{T-t} \times N(D_a + \sigma \sqrt{T-t}) - X e^{-r(T-t)} N(D_a)
\]

(10)

\[
\text{Put (Asian)} = -S_0 e^{-r(T-t)} \sigma \sqrt{T-t} \times N(-D_a + \sigma \sqrt{T-t}) - X e^{-r(T-t)} N(-D_a)
\]

(11)

where \((T-t)\) is the lifetime of the option and

\[
D_a = \frac{\ln(S/X) + (r - q - \sigma^2/2) \frac{T-t}{2}}{\sigma \sqrt{(T-t)/3}}
\]

Exact pricing formulae for Asian options using the arithmetic average do not exist. However, approximate forms such as Monte Carlo techniques can be used. A fruitful area for further research is the implicit premium on the insurance inherent in these options.

Application:

Consider the electricity transmission and distribution company described earlier in the section on binary options. The company may prefer to hedge on the basis of average spot electricity prices rather than the "hit or miss" nature of the binary option. If the firm buys an Asian call option it reduces its profit margin by a fixed

\[ \text{profit margin} = \text{profit margin}_\text{without option} - \text{profit margin}_\text{with option} \]

where the profit margin with option is calculated using the formula above.
amount in exchange for reduced volatility in the price it pays for power.

**Barrier Options**

Barrier options, also called *touch options*, have payoffs which depend not only on the price of the underlying asset at expiration but also whether or not the option has passed through or touched some trigger point known as a barrier. A down-and-out option *Figure 8A* automatically becomes worthless when the stock price falls below some predetermined barrier price, denoted by $B_l$ in the diagram. Similarly, a down-and-in option *Figure 8B* does not provide a payoff unless the stock price falls below some barrier price, denoted by $B_u$, at least once during the life of the option. These options are also referred to as *knock-out* and *knock-in* options. A barrier option that becomes worthless if an event occurs is also called an *exploding option*.

**Figure 8**

**Payoff Diagrams**

There are various forms of barrier options. Some require touching the barrier only once, while others require two or three "strikes". The latter type is called a baseball option for obvious reasons. *Figure 9* shows the payoff for an up-and-out call options. This option has the property that the maximum payoff is determined by the barrier $B_l$; above $B$ the option is worthless.

**Pricing Formula for Barrier Options**

Pricing formulae for barrier options are presented by McDonald. In the most trivial case, where the stock price does not cross the barrier over the life of the option, the option is priced as a simple call or put option, according to the Black-Scholes formula. If the stock price stays below the barrier price $B_l$, the price for an up-and-out call option is given by

$$
C_{up \text{in}} = S_0 e^{-\alpha T} \left[ N(d_3) + \left(\frac{e^{\alpha T}}{2}\right)^{d_3} \right] x \left( N(d_u) - N(d_l) \right)
\quad - X e^{-\alpha T} \left[ N(d_4) + \left(\frac{e^{\alpha T}}{2}\right)^{d_4} \right] x \left( N(d_u) - N(d_l) \right)
$$

(12)

where the arguments $d_3$ through $d_8$ are defined according to:

...
For an up and out call the price is simply given by the price of a European call option minus the value given by Equation 12. Pricing formulae for other types of barrier option are similar. For a down and in call option \((S>H)\), the price is

\[
C_{\text{down/in}} = S e^{-q(T-t)} \left( \frac{H}{S} \right)^{\frac{\sigma^2 T}{2}} N(d_3) - X e^{-r(T-t)} \left( \frac{H}{S} \right)^{\frac{\sigma^2 T}{2}} \times N(d_4)
\]

(13)

For an up and in put (for \(S<H\)) the price is

\[
P_{\text{up/in}} = -S e^{-q(T-t)} \left( \frac{H}{S} \right)^{\frac{\sigma^2 T}{2}} \times N(-d_3) + X e^{-r(T-t)} \left( \frac{H}{S} \right)^{\frac{\sigma^2 T}{2}} \times N(-d_4)
\]

(14)
Finally, for a down and in put (for \( S_t > B \)) the price is
\[
P_{\text{down/in}} = -e^{-r(T-t)} \left[ N(-d_2) - N(-d_3) \right] + e^{-r(T-t)} \left[ N(d_2) - N(d_3) \right],
\]
(15)

Consider an option with the following properties:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exercise Price</td>
<td>$100</td>
</tr>
<tr>
<td>Volatility</td>
<td>30.00%</td>
</tr>
<tr>
<td>Risk-free interest rate</td>
<td>8.00%</td>
</tr>
<tr>
<td>Time to Expiration (years)</td>
<td>0.5</td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>0.00%</td>
</tr>
<tr>
<td>Barrier</td>
<td>$125</td>
</tr>
</tbody>
</table>

The prices of the barrier options are thus:

<table>
<thead>
<tr>
<th></th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black-Scholes</td>
<td>$10.39</td>
<td>6.47</td>
</tr>
<tr>
<td>Up &amp; In</td>
<td>$8.48</td>
<td>0.12</td>
</tr>
<tr>
<td>Up &amp; Out</td>
<td>$1.90</td>
<td>6.35</td>
</tr>
<tr>
<td>Down &amp; In</td>
<td>$10.39</td>
<td>6.47</td>
</tr>
<tr>
<td>Down &amp; Out</td>
<td>$0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

For comparison, the price of the vanilla call and put options, calculated from the Black-Scholes formula, are also shown. Note that the prices of both down and in options are the same as the plain options since the stock price is below the barrier.

Application

Insurance companies are in the business of accepting risk others do not want. They, in turn, lay off risk they do not want through contracts with reinsurance firms. Catastrophic events such as multiple hurricanes or the September 11, 2001 terrorist attacks in the U.S.A. result in massive payouts for the reinsurance firms. Some analysts believe the insurance industry would have been unable to survive following a second incident of the magnitude of September 9/11 attacks. It is possible to spread this risk throughout the financial marketplace via a barrier option, perhaps a "touch twice and in" call option based on some measures of aggregate insurance claims in a time period*.

**Forward Start Option**

A forward start option is an option which has a deferred start date. With plain vanilla options, the option becomes valid immediately; with the forward start option there is a time delay before the option becomes activated. The value of the forward start option is given by \( ce^{-rT_1} \), where \( T_1 \) is the option start time and \( c \) is the value at time zero of an at-the-money option with lifetime \( T_2 - T_1 \). An alternate formulation, due to Zhang14, gives the value of a European forward start call option as
\[
C_{FS} = S\left[ e^{-r(T_2-T_1)} N(d_{1B}) - e^{-r(T_2-T_1)} N(d_{2B}) \right],
\]
(16)

where
\[
d_{1B} = \frac{1}{2} \left[ \ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T_2 - T_1) \right] \sqrt{T_2 - T_1}, \quad d_{2B} = d_{1B} + \sigma \sqrt{T_2 - T_1},
\]

The value of a European forward start put option is, similarly,
\[
P_{FS} = -S\left[ e^{-r(T_2-T_1)} N(-d_{1B}) - e^{-r(T_2-T_1)} N(-d_{2B}) \right],
\]
(17)

As an example, the prices of the forward start call and put options are calculated when the time to maturity is one year, the spot underlying asset price is $75, volatility is 25 per cent, the interest rate is 10 per cent, the yield (q) on the underlying asset is 5 per cent, and the forward start date is six months in the future. Equations 16 and 17 indicate the prices of the forward start call and forward start put options are $5.89 and $4.13 respectively.

**Application**

We often find forward start options associated with the swaps market. A firm might be considering a deferred...
start interest rate swap, which is simply an ordinary swap where the first difference check is not remitted until further into the future than normal. A swaption is an option to enter into a swap and, may be either a put (giving its owner the right to pay the floating rate) or a call (giving its owner the right to pay the fixed rate). A swaption on a deferred start swap is therefore an option with a delayed starting date. As with a European option, even though it cannot be exercised immediately, it does have value as shown earlier in the valuation equations.

Conclusions

Second generation options have become more prevalent in the financial markets over the last several years. This paper has sought to enhance understanding of these options through an examination of the payoff/profit diagrams, the pricing equations, and potential managerial applications. This exposition should make it easier for instructors to include exotic options in their courses, thereby improving the introductory derivatives course.

REFERENCES

4. Strong, R., Derivatives: An Introduction (South-Western, 2002)