PRACTICAL SOLUTION USING MARKOV CHAINS TO DESCRIBE OPTION STATES

C. G. DEACON, Ph.D., M.B.A.
Faculty Member
and
PROFESSOR A. J. FASERUK, Ph.D.
Faculty of Business Administration
Memorial University of New Foundland
St. John's, CANADA.

Abstract

This paper demonstrates how Markov Chain analysis can be employed to describe the behavior of an American option which can exist in one of several states during its lifetime. The option states “in the money” and “out of the money” are considered to be transient states, while exercising the option or letting the option expire are considered to be absorbing. The transition probabilities that outline the movement of an option between these states are explained in terms of steady state and time dependent probabilities.

Key words: Markov Chain; American Option; Transition Probabilities.
JEL Classification: C23, C53, G13

Introduction

One of the most important decisions to be made after an option is purchased is what to do subsequently with it*. The option may be exercised if it is trading in the money. As opposed to exercise, there is always the ability to sell the option, thereby realizing both the intrinsic and time values. Conversely, the investor may choose to hold it longer in anticipation of future increases in the value of the option as early exercise terminates the time premium, or extrinsic value, of the option. Holding it for too long a period may, however, be detrimental to value maximization if the option were to decline in value and if it subsequently begins to trade out of the money. The worst case scenario is, of course, when the option is held to maturity and expires worthless.

Given that there are several states of nature or conditions in which the option can exist, it is possible to ascertain the probability of moving from one state to another and, therefore, to apply the mathematics of Markov chains to an option. If the probability of moving from one state to another is known, Markov analysis can be used to predict the behavior of the system over time. The schematic model below describes the option’s states of being in or out of the money and the three actions which the investor may undertake. The reader will note that this model is appropriate for either call or put options as the states are in or out of the money.

Model

Duan* has used Markov chains to study American option pricing. The current study considers, specifically, the different states of the option, and employs Markov analysis to describe the probability that the system will move between these states. The option may be considered to exist in five states, as shown in Figure 1.

* While an option may also trade at the money, it is also realized that an option in this state has no intrinsic value. Accordingly, these options are considered to be out of the money when \( S = X \), where \( S \) is the stock price and \( X \) is the option strike price. Thus a call option will be in the money when \( S > X \), but out of the money when \( S \leq X \). Conversely, a put option will be in the money when \( S < X \) and out of the money when \( S \geq X \).

The authors own full responsibility for the contents of the paper.
"out of the money"; the option will always be in one of these states unless exercised or sold. There are also three absorbing states, "exercise the option", "sell the option" and "let the option expire". The last of these will not be considered, since the only time that an option will be allowed to expire worthless will be on the expiry date, and only if it is out of the money. Accordingly, the model reduces to four states, two transient and two absorbing, as shown in Figure 2. The arrows indicate the allowed transitions between states.

In Figure 2, state 1 is an investor exercising the option, state 2 is the investor selling the option, state 3 is the option being in the money and state 4 is the option being out of the money.

The state transition probabilities, $p_{ij}$, represent the probability of the option moving from state $i$ to state $j$ in the next time interval. Since states 1 and 2 are absorbing, $p_{11} = p_{22} = 1$. 
For an option which begins in state 3 (in the money), the possible transition probabilities are:

- \( P_{31} \) - exercise the option,
- \( P_{32} \) - sell the option while in the money,
- \( P_{33} \) - the probability that the option stays in the money (holding state),
- \( P_{34} \) - the probability of an in the money option being out of the money in the next time interval.

In Markov analysis, the transition probabilities for a given beginning state of the system must be mutually exclusive, collectively exhaustive and must sum to one. For an option beginning in state 3, the relevant probabilities are:

\[
P_{31} + P_{32} + P_{33} + P_{34} = 1
\]

Similarly, the transition probabilities associated with state 4 for an option beginning out of the money are:

\[
P_{42} + P_{43} + P_{44} = 1
\]

Note that \( p_{41} = 0 \) because an option that is out of the money at expiry will not be exercised. The complete transition probability matrix is, therefore,

\[
\begin{bmatrix}
P_{31} & P_{32} & P_{31} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
0 & P_{43} & P_{43} & P_{44}
\end{bmatrix}
\]

Since states 1 and 2 are absorbing, the matrix becomes

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
P_{31} & P_{32} & P_{33} & P_{34} \\
0 & P_{43} & P_{43} & P_{44}
\end{bmatrix}
\]

and the steady state probabilities are calculated from

\[
F = (I - Q)^{-1} \times R
\]

where the inverse of the I-Q matrix gives the expected number of transitions between states 3 and 4 that the option will undergo before being absorbed in states 1 or 2. Solving for the inverse matrix yields

\[
(I - Q)^{-1} = \begin{bmatrix} N_{33} & N_{34} \\ N_{43} & N_{44} \end{bmatrix}
\]

This matrix implies that if the option starts in the money (state 3), it should be expected to spend \( N_{33} + N_{34} \) periods in states 3 or 4 before being absorbed. Similarly, if the option starts in state 4, it should spend \( N_{43} + N_{44} \) periods in states 3 or 4 before being absorbed. We can also write the above matrix as

\[
\begin{bmatrix} \Phi_{31} & \Phi_{32} \\ \Phi_{41} & \Phi_{42} \end{bmatrix}
\]

where \( \Phi_{31} \) and \( \Phi_{41} \) represent the probabilities of the option being exercised, given its initial state (in or out of the money) and \( \Phi_{41} \) and \( \Phi_{42} \) are the probabilities of the option being sold before the expiry date.

In the absence of any specific information about the various probabilities, the first approach to the problem is to use the Laplace criterion and assume that the probabilities associated with a particular state are equally likely and constant. If there are four outcomes, then the probability of each occurring is \( \frac{1}{4} \). Thus, \( p_{31} = p_{32} = p_{33} = p_{34} = \frac{1}{4} \) and similarly, \( p_{43} = p_{43} = \frac{1}{4} \). Using the procedure outlined above yields

\[
R = \begin{bmatrix} p_{31} & p_{32} \\ 0 & p_{42} \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/3 \end{bmatrix}
\]

and

\[
Q = \begin{bmatrix} p_{33} & p_{34} \\ p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/3 & 1/3 \end{bmatrix}
\]

Thus

\[
(I - Q)^{-1} = \begin{bmatrix} 8/5 & 3/5 \\ 4/5 & 9/5 \end{bmatrix}
\]

and

\[
(I - Q)^{-1} R = \begin{bmatrix} 2/5 & 3/5 \\ 1/5 & 4/5 \end{bmatrix}
\]
Hence \( \Phi_1 = 0.4, \Phi_2 = 0.6, \Phi_3 = 0.2 \) and \( \Phi_4 = 0.8 \). Thus, for an American option that begins in the money, there is a 40 percent probability that it will be exercised early, and a 60 percent probability that it will be sold before expiry. Also, when the option starts out of the money, there is a 20 per cent chance that it will be exercised early and a 80 per cent probability that it will be sold before expiry. These results are summarized in Table 1 below:

**Table 1**

<table>
<thead>
<tr>
<th>( P_{31} )</th>
<th>( P_{32} )</th>
<th>( P_{33} )</th>
<th>( P_{34} )</th>
<th>( P_{41} )</th>
<th>( P_{42} )</th>
<th>( \Phi_{31} )</th>
<th>( \Phi_{32} )</th>
<th>( \Phi_{41} )</th>
<th>( \Phi_{42} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
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<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.4</td>
<td>0.4</td>
<td>0.1</td>
<td>0.45</td>
<td>0.45</td>
<td>0.366</td>
<td>0.633</td>
<td>0.3</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>0.45</td>
<td>0.45</td>
<td>0.05</td>
<td>0.475</td>
<td>0.475</td>
<td>0.35</td>
<td>0.65</td>
<td>0.316</td>
</tr>
</tbody>
</table>

The above analysis has shown that, by assuming the transition probabilities to be constant, the long term behavior of the option can be studied up to the day of expiry. At expiry, the probability of exercising the option is automatically one or zero, depending on whether the option is in or out of the money. The assumption of constants in the analysis of options has been used in the past, particularly in the derivation of the Black-Scholes option pricing model; both the volatility, \( \sigma \), of the underlying asset and the risk free rate, \( r \), are assumed to be constant throughout the life of the option, even though it is known that the volatility may increase for large deviations from the strike price. The risk free rate may also vary considerably over the life of the option. A more mathematically correct model will take into account the fact that the probabilities can and do change with time. The subject of time dependent probabilities is discussed in the following paragraphs.

**Time Dependent Probabilities**

The transition probabilities that describe the option’s moving between states are time dependent, since the option owner’s decision to sell or exercise the option will be influenced by the time to expiry. The seven probabilities which have been used to describe the possible transitions from states 3 and 4 are calculated as follows:

- \( P_{31} \): A option that is in the money on the day of expiry will be exercised with probability \( P_{31} = 1 \). Before expiry, \( P_{31} = 0 \) for a European option, by \( 0 < P_{31} < 1 \) for an American option if exercised early. Assume that \( P_{31} \) depends on the stock price, \( S \), and its standard deviation, \( d \), which can be calculated from historical data using the relation:

\[
d^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( S_i - \frac{1}{n} \sum_{i=1}^{n} S_i \right)^2
\]

A possible functional form for \( P_{31} \) is

\[
P_{31} = e^{-d^2}
\]

The exponential function is commonly used in probability theory. Here it is employed because the function shows that the transition probability has a maximum value when \( d = 0 \) and decreases as the deviation increases from zero, as shown in Figure 3.

* Theta for a call option is given by \( \Theta = \frac{S e^{-rT} d^2}{2 \sqrt{2\pi}} - r S e^{-rT} \phi(d) \).
FIGURE 3
PROBABILITY OF EARLY EXERCISE DEPENDS ON THE STANDARD DEVIATION OF THE UNDERLYING ASSET PRICE

- \( p_{32} \): This is the probability of an in the money option being sold before its expiry date. This will depend on the current value of the option, \( C \), the difference between the strike price and the current stock price, \( S-X \), and how quickly the option loses value. This is described by the greek letter \( \theta \), which is a measure of the sensitivity of a call option to the time remaining until expiry (Strong)\(^8\). High values of \( C \) and \( S-X \) suggest that the option is likely to be sold, whereas a high value of \( \theta \) suggests the option loses value rapidly and that a sale is unlikely as the option approaches maturity. This probability is written as

\[
p_{32} = C (S-X)e^{-\theta}
\]

The value of the option can be obtained from the binomial option pricing model (Hull)\(^9\), i.e.,

\[
C = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \times \text{max} (0, Su_i d^{n-i} - X)
\]

where the symbols have their usual meaning. The “up” and “down” probabilities can be estimated from historical data.

- \( p_{33} \): This is the probability of the owner holding onto the option while still in the money. Mathematically, this will be the probability that remains once the other transitions from state 3 have been considered. Hence

\[
p_{33} = 1-p_{31} - p_{32} - p_{34}
\]

- \( p_{34} \): This is the probability of the option switching from “in the money” to “out of the money”. This will depend on the absolute value of \( S-X \); the smaller the difference between the strike price and stock price, the more likely the option will switch states. \( p_{34} \) will also depend on the volatility of the underlying stock, \( \sigma \). A possible functional form for the probability is

\[
p_{34} = \sigma \exp \left( \frac{(X-S)^2}{2d^2} \right)
\]

where \( d \) is the standard deviation. \( p_{43} \) will have nearly identical functional form.

- \( p_{43} \): This is the probability of an out of the money option being sold. The probability function will be similar to \( p_{32} \) except that a sale will depend on the value of \( X-S \) and will also be more likely with increasing \( \theta \). Thus,

\[
p_{43} = C(X-S)e^{-\theta}
\]

The intraday probabilities will change during the life of the option. However, if each of the \( p_{ij} \)'s are averaged over the life of the option, the matrix \( F \) becomes

\[
\begin{bmatrix}
0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 \\
\end{bmatrix}
\]

and hence it is possible to calculate the probability that an option will either be exercised or sold, given its initial state. For example, using the probabilities given in the top row of Table 1,

\[
\Phi_1 = \frac{p_{31} (1-p_{33})}{1-p_{31} + p_{32} + p_{33}}
\]

\[
= \frac{0.25 \times (1 - 0.333)}{1 - 0.25 - 0.25 \times 0.333 - 0.333 + 0.25 \times 0.333}
\]

\[
= 0.60
\]

\( \Phi_2, \Phi_3 \) and \( \Phi_4 \) may be calculated similarly to give the values shown in the table.
Conclusions

This paper has described how the mathematics of Markov Chains can be applied to the various states in which options can be found during their lives. If the transition probabilities between the various states are constant, then the steady state probabilities that the option will either be exercised or allowed to expire can be calculated using standard matrix algebra. In actuality, the transition probabilities depend on the volatility of the underlying asset and will change as the time to expiry decreases. However, by assuming an average value for each probability, then an estimate of the steady state probabilities can be obtained.

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