Aharonov-Bohm Flux and the Tunneling Density of States of the Calogero-Sutherland Model on a Ring

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We present a detailed discussion of the many-body states of the Calogero-Sutherland model. The main result is an exact calculation of the point contact tunneling amplitude obtained using the Calogero-Sutherland model, which is equivalent to that obtained using chiral Luttinger liquid theory.

1 Introduction

The Calogero-Sutherland (CS) model is an exactly solvable Hamiltonian which describes a system of electrons on a ring interacting by a $1/r^2$ potential[1,2]. It is closely related to the Luttinger-Thirring model which used to describe the real physical system of edge states in the fractional quantum Hall (FQH) effect[3]. The usefulness of the CS model lies with its exact formulation on finite systems, which allows one to perform exact calculations.

In the following section we introduce the CS model and describe its solutions, paying particular attention to the difference between even and odd numbers of particles. We discuss the effect of introducing a magnetic flux in the system. In Section 3 we present an exact result for point contact tunneling amplitude using the many-body ground state of the CS model.

2 The Calogero-Sutherland Model

2.1 The Hamiltonian

A system of N interacting electrons on a ring may be described by the CS Hamiltonian,

$$H = \frac{-1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\pi^2}{L^2}\sum_{j<l}^{N}\frac{\lambda(\lambda-1)}{\sin^2\frac{\pi(x_j-x_l)}{L}} \tag{1}$$

where x_i is the position of the ith electron, L is the circumference of the ring and λ is the interaction strength. Here we consider the cases where $\lambda \geq 1$; the system is non-interacting when $\lambda = 1$. The mass is set to unity. It is convenient to rewrite (1) in terms of the dimensionless variables $z_j \equiv \exp\left(\frac{i2\pi x_j}{L}\right)$. Then H takes the form

$$H = \left(\frac{2\pi}{L}\right)^2 h \equiv \left(\frac{2\pi}{L}\right)^2 (h_0 + h_1) \tag{2}$$

$$h_0 = \frac{1}{2}\sum_{j=1}^{N}\left(z_j \frac{\partial}{\partial z_j}\right)^2 \tag{3}$$

$$h_1 = -\lambda(\lambda-1)\sum_{j<l}^{N}\frac{z_l z_j}{(z_j - z_l)^2} \quad . \tag{4}$$

2.2 The Ground State

For a general λ the ground state is given by [1] (see Appendix A for more details)

$$\Psi_0^\lambda(z_1, \dots, z_N) = \prod_{i<j}(z_i - z_j)^\lambda \prod_k z_k^{J_0} \tag{5}$$

It is evident that when λ is even (odd) integer the ground state is bosonic (fermionic) and a non-integral λ yields anyonic solutions. We will only consider the cases for λ odd. Then, in order to satisfy periodic boundary conditions J_0 must be an integer,

$$J_0 = \begin{cases} -\frac{\lambda(N-1)}{2} & N \text{ odd} \\ -\frac{\lambda(N-1)}{2} \pm \frac{1}{2} & N \text{ even} \end{cases} \tag{6}$$

and the eigenvalue of h is

$$e_0 = \begin{cases} \frac{N\lambda^2(N^2-1)}{24} & N \text{ odd} \\ \frac{N(\lambda^2(N^2-1)+3)}{24} & N \text{ even} \end{cases} \quad . \tag{7}$$

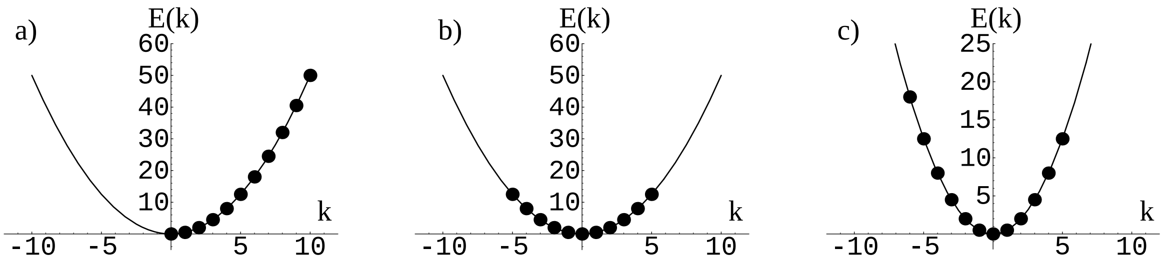

Figure 1: (a) The energies of the occupied single particle states in the non-interacting many-body state $\prod_{i<j}(z_i - z_j)$. (b) The ground state for an odd number of particles: it is obtained by boosting the momentum of each particle in (a) by $J_0 = -\frac{(N-1)}{2}$, corresponding to the factor $\prod_k z_k^{J_0}$ in Eq. 5. (c) The ground state for an even number of particles. It is doubly degenerate since the last particle may by placed at $k = -N/2 + 1$ (as shown in the plot) or at $k = N/2 - 1$.

As an example we consider the ground state of N non-interacting particles. In this case the single particle solutions of h_0 are given by $\Psi_k(z) = z^k$ with energy $k^2/2$, and to satisfy the periodic boundary conditions k must be an integer. The many-body state must satisfy the additional condition that the wavefunction be antisymmetric under exchange of particles. The ground state may be constructed by adding particles to the lowest energy single particle states until N states are filled and then antisymmetrising the result (see Figs. 1b and 1c).

Equivalently, one may construct the state $\prod_{i,j}(z_i - z_j)$ shown in Fig. 1a and then boost each particle by $J_0 = -\lambda\frac{(N-1)}{2}$ to obtain Fig. 1b. The double degeneracy of the states with momenta $\pm k$ implies that many-body states with even N are doubly degenerate, while those with odd N are unique. The energy of the ground state is $\sum_k \frac{k^2}{2}$ which is $\frac{N(N^2-1)}{24}$ for N odd and $\frac{N(N^2+2)}{24}$ for N even. Comparing to the energy of the interacting case this one may infer that the momenta associated with interacting states are λk.

2.3 Excited States

Now we follow the procedure given by Ha[4] to find the excited states of the system. Let $\Psi_n \equiv \Psi_0 P_n^\lambda$ be an eigenstate with eigenvalue e_n. The functions P_n^λ satisfy the eigenvalue equation

$$\tilde{h}P_n^\lambda = (\tilde{h}_0 + \lambda\tilde{h}_1)P_n^\lambda = \tilde{e}_n P_n^\lambda \tag{8}$$

where the eigenvalue $e_n = e_0 + \frac{1}{2}\tilde{e}_n$ and

$$\tilde{h}_0 = \sum_{j=1}^{N}\left(z_j\frac{\partial}{\partial z_j}\right)^2 \quad \left[\pm\sum_{j=1}^{N} z_j\frac{\partial}{\partial z_j}\right] \tag{9}$$

$$\tilde{h}_1 = \sum_{j<k}^{N}\left(\frac{z_j+z_k}{z_j-z_k}\right)\left(z_j\frac{\partial}{\partial z_j} - z_k\frac{\partial}{\partial z_k}\right) \tag{10}$$

where the second term of (9) appears only for N even and the sign depends on the choice of ground state (see Appendix B). The solutions P_n^λ are Jack polynomials (see Appendix E), but we find it more convenient to follow Ha's procedure and express the Hamiltonian $\tilde{h}$ in terms of a basis of symmetric polynomials $\Phi_{\mathbf{n}}$. These are defined as

$$\Phi_{\mathbf{n}}(z_1,\ldots,z_N) = \sum_P \prod_{j=1}^{N} z_j^{n_{P_j}} \tag{11}$$

where $\mathbf{n} = (n_1,\ldots,n_N)$ and $n_1 \geq n_2 \geq \ldots \geq n_N$. The sum is over *all* permutations of the integers $n_1,\ldots,n_N$, therefore generally there are $N!$ terms in the sum. In particular, $\Phi_0 = N!$. We find that (see Appendix C)

$$\tilde{h}_0\Phi_{\mathbf{n}} = \sum_{j=1}^{N}(n_j)^2\Phi_{\mathbf{n}} \quad \left[\pm\sum_{j=1}^{N} n_j\Phi_{\mathbf{n}}\right] \tag{12}$$

$$\tilde{h}_1\Phi_{\mathbf{n}} = \sum_{j<k}^{N}(n_j - n_k)\sum_{s=0}^{n_j-n_k-1}\Phi_{(n_1,\ldots,n_j-s,\ldots,n_k+s,\ldots,n_N)} \tag{13}$$

The states which have no n_i, n_j such that $n_i - n_j > 1$ will only get the $s = 0$ term in the sum of Eq. 13 therefore they are eigenstates of $\tilde{h}$. Some examples of eigenstates are $\Phi_{(0,0,0,0)}$, $\Phi_{(1,0,0,0)}$ and $\Phi_{(1,1,0,0)}$. As another example we consider the eigenstates shown in Fig. 2. The action of $\tilde{h}$ on the symmetric polynomials contributing to each state is listed in Table 1. Fig. 2a shows one of the ground states Ψ_0 of a four particle non-interacting system. Its eigenvalue is determined by Eq. 7 and it is $e_0 = 3$. Fig. 2b shows an excited eigenstate $\Psi_0\Phi_{(1,0,0,0)}$ with eigenvalue $e_0 + \frac{\tilde{e}}{2} = \frac{11}{2}$. The states $\Psi_0\Phi_{(1,1,0,-1)}$ and $\Psi_0\Phi_{(2,0,0,-1)}$ are not eigenstates since $\tilde{h}$ mixes the symmetric polynomial with other states. Diagonalising $\tilde{h}$, whose components are listed in Table 1, yields the linear combinations which are the eigenstates shown in Figs. 2c and 2d.

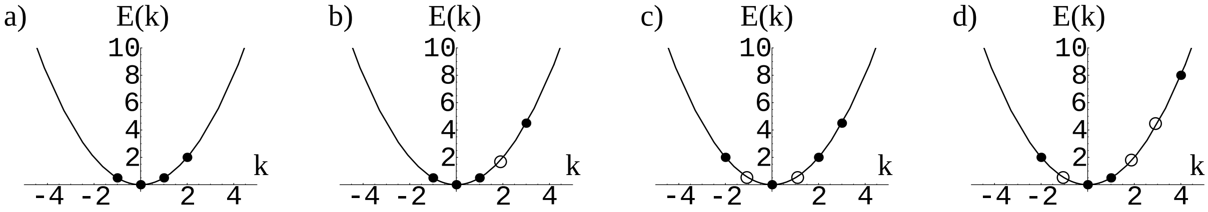

Figure 2: The energies of some eigenstates of the non-interacting four particle system. The energy of each state shown is calculated by summing the energy of each filled dot. (a) The ground state $\Psi_0\Phi_{(0,0,0,0)}$ with energy $e_0 = 3$. (b) The state $\Psi_0\Phi_{(1,0,0,0)}$ with energy $e_0 + \frac{\tilde{e}}{2} = \frac{11}{2}$. (c) The state $\Psi_0\Psi_{1,1,0,-1} + \frac{2}{3}\Psi_0\Phi_{(1,0,0,0)}$ with energy $e_0 + \frac{\tilde{e}}{2} = \frac{17}{2}$. (d) The state $\Psi_0\Phi_{(2,0,0,-1)} + \Psi_0\Phi_{(1,1,0,-1)} + \Psi_0\Phi_{(1,0,0,0)}$ with energy $e_0 + \frac{\tilde{e}}{2} = \frac{21}{2}$.

	$\Phi_{(0,0,0,0)}$	$\Phi_{(1,0,0,0)}$	$\Phi_{(1,1,0,-1)}$	$\Phi_{(2,0,0,-1)}$
$\tilde{h}_0$	0	$2\Phi_{(1,0,0,0)}$	$4\Phi_{(1,1,0,-1)}$	$6\Phi_{(2,0,0,-1)}$
$\tilde{h}_1$	0	$3\Phi_{(1,0,0,0)}$	$7\Phi_{(1,1,0,-1)} + 4\Phi_{(1,0,0,0)}$	$9\Phi_{(2,0,0,-1)} + 4\Phi_{(1,1,0,-1)} + 6\Phi_{(1,0,0,0)}$

Table 1: The action of $\tilde{h}$ on the four symmetric polynomials appearing in the eigenstates illustrated in Fig. 2. Note that $|\mathbf{n}| \equiv \sum_j n_j$ is unchanged in each of the final states.

2.4 Aharonov-Bohm Flux and Current

The current J of the state Ψ is given by $J\Psi = \sum_j z_j \frac{\partial}{\partial z_j}\Psi$. For the ground state with odd N we have $J = 0$, while for even N it is $J = \pm 1$. For a general excited state $\Psi_0\Phi_{\mathbf{n}}$ the change in momentum with respect to the ground state is $\sum_i n_i$. In particular, $\Psi_0\Phi_n \equiv \Psi_0\Phi_{(n,\ldots,n)}$ is an eigenstate of the system in which each particle has its momentum boosted by n.

Now we consider the effect of threading a Bohm-Aharonov flux through the ring. For m flux quanta the gauge field $A_\theta = \frac{m\phi_0}{L}$ is added to the Hamiltonian, which then may be "gauged away" by adding an appropriate phase to the wavefunction. The required phase is $\sum_j \frac{m 2\pi x_j}{L}$, that is the wavefunction gets multiplied by $\exp\left(i \sum_j \frac{m2\pi x_j}{L}\right) \equiv \prod_j z_j^m$, and the boundary conditions are modified accordingly. In general this corresponds to a total boost of Nm, and when m is an integer this corresponds to a transformation of the wavefunction $\Psi_0\Phi_{\mathbf{n}} \to \Psi_0\Phi_{\mathbf{n}+m}$ where $\mathbf{n} + m = (n_1 + m, \ldots, n_N + m)$.

As a function of the flux m the ground state evolves as a periodic function of m with period one. When N is odd the ground state energy in the interval $m \,\epsilon\, [-\frac{1}{2}, \frac{1}{2}]$ is given by $e_0 + N\frac{m^2}{2}$, while for N even and $m \,\epsilon\, [0, 1]$ the ground state energy is $e_0 + N\frac{(m-\frac{1}{2})^2 - \frac{1}{4}}{2}$, where e_0 is given by Eq. 7. The energy of the even N state is actually lowered by the addition of flux, with a minimum at $m = \frac{1}{2}$ and current $J = 0$.

2.5 Normalisation

In general the normalisation of the basis $\Psi_0\Phi_{\mathbf{n}}$ is a tricky task (see Appendix E). For the purposes of this paper we need only the normalisation of the N-particle ground state, which is given by

$$\langle N_0^\lambda | N_0^\lambda \rangle = \int_0^L \frac{dx_1 \ldots dx_N}{L^N} |\Psi_0^\lambda(x_1, \ldots, x_N)|^2 \tag{14}$$

$$= \frac{1}{(2\pi i)^N} \oint_{|z|=1} \frac{dz_i \ldots dz_N}{z_1 \ldots z_N} \prod_{i<j}^{N} (z_i - z_j)^\lambda \left(\frac{1}{z_i} - \frac{1}{z_j}\right)^\lambda. \tag{15}$$

Here we have converted the line integral over x to a contour integral around the circle $|z| = 1$. It is evident that the only non-zero contribution to the integral comes from the coefficient of unity in the product. The result is (see Appendix E)

$$\langle N_0^\lambda | N_0^\lambda \rangle = \frac{(\lambda N)!}{(\lambda !)^N} = \binom{\lambda N}{\lambda} \langle N - 1_0^\lambda | N - 1_0^\lambda \rangle \tag{16}$$

This result may be generalised for any λ by writing it in terms of gamma functions.

3 Tunneling Density of States

The low energy dynamics of a fractional quantum Hall (FQH) system occur at its edges because it is only here where gapless excitations exist, thanks to a spatially dependent confining potential which crosses the Fermi energy. However, it is still the bulk which determines the properties of the edge states. Hence the parameters of the chiral Luttinger theory which describes the edge states are derived from properties of the bulk, namely the correlation exponents of the Luttinger theory are related to the bulk filling fraction $\nu = n_e/B$ of the FQH system.

The edge state of the FQH system for $\nu = 1/m$ is described by a chiral Luttinger liquid model [5], whose ground state many-body wavefunction and correlation exponents may be derived from the bulk FQH state [6]. The many-body ground state of the CS model presented above resembles the ground state of the Luttinger model and has the same correlation exponents [7]. Moreover, at low energies the CS model may be written as a Luttinger-Thirring model [3]. In this section we compare the result of a calculation using the CS model with the same quantity calculated using chiral Luttinger liquid theory.

Here we are interested in computing the quantity $\langle N + 1_0^\lambda | \psi^\dagger(0) | N_0^\lambda \rangle$, where $\psi^\dagger(0)$ is a field operator which adds a single particle to the point $x = 0$ and the ket $|N_0^\lambda\rangle$ is the N particle ground state. It is convenient to expand $\psi(x)$ in terms of standing waves

$$\psi(x) = \sum_k a_k \frac{\exp\left(\frac{i2\pi k x}{L}\right)}{\sqrt{L}} = \sum_k a_k \frac{z^k}{\sqrt{L}} \tag{17}$$

where a_k is the destruction operator for a particle with momentum k. The real space represention of $a_k^\dagger |N\rangle$ is

$$a_k^\dagger \psi(z_1, \ldots, z_N) = (N+1)^{-1/2} A_{N+1} \left\{ z_{N+1}^k \psi(z_1, \ldots, z_N) \right\}. \tag{18}$$

Here A_{N+1} represents the antisymmetrisation operator which acts on $z_1 \ldots z_{N+1}$ and the factor $(N+1)^{-1/2}$ is due to the normalisation of the antisymmetrised state. Then $\psi^\dagger(0)|N_0^\lambda\rangle$ may be represented by the $N+1$-particle wavefunction

$$(N+1)^{-1/2} A_{N+1} \left\{ \Psi_0^\lambda(z_1, \ldots, z_N) \sum_{k=-C^-}^{C^+} \frac{z_{N+1}^k}{\sqrt{L}} \right\}.$$

The sum is equivalent to $\delta(x_{N+1})$ in the limit $C^\pm \to \infty$, so the $(N+1)$th particle is indeed added at the point $x = 0$. The procedure for performing the antisymmetrisation operation involves writing down all of the terms inside the parentheses and then adding all the terms which are obtained by exchanging $z_i \leftrightarrow z_{N+1}$.

The tunneling amplitude is proportional to the matrix element $\langle N + 1^\lambda | \psi(0)^\dagger | N^\lambda \rangle$. We compute this overlap at $T = 0$ (see Appendix D),

$$\frac{\langle N + 1_0^\lambda | \psi(0)^\dagger | N_0^\lambda \rangle}{|\langle N + 1_0^\lambda | N + 1_0^\lambda \rangle \langle N_0^\lambda | N_0^\lambda \rangle|^{1/2}} \approx \frac{1}{|\langle N + 1_0^\lambda | N + 1_0^\lambda \rangle \langle N_0^\lambda | N_0^\lambda \rangle|^{1/2}} \frac{(N+1)^{1/2} \langle N_0^\lambda | N_0^\lambda \rangle}{L^{1/2}}. \tag{19}$$

This result does not depend on the cutoffs, provided $C \gtrsim \lambda N/2$. Nor does the result change when an integer amount of flux is added to the system, that is $\langle N+1^\lambda_m|\psi(0)^\dagger|N^\lambda_m\rangle = \langle N+1^\lambda_0|\psi(0)^\dagger|N^\lambda_0\rangle$ where m is the number of flux quanta and $|N^\lambda_m\rangle$ is the boosted state which is obtained after a gauge transformation that eliminates the flux. However, when m is non-integral the result becomes (assuming N even)

$$\frac{\langle N+1^\lambda_0|\psi(0)^\dagger|N^\lambda_0\rangle}{|\langle N+1^\lambda_0|N+1^\lambda_0\rangle\langle N^\lambda_0|N^\lambda_0\rangle|^{1/2}} \approx \left[\frac{\langle N^\lambda_0|N^\lambda_0\rangle(N+1)}{\langle N+1^\lambda_0|N+1^\lambda_0\rangle L}\right]^{1/2} \frac{1}{\pi}\sum_{j=-C}^{C}\frac{\sin\pi(\lambda j+\lambda N/2-m)}{\lambda j+\lambda N/2-m}. \quad (20)$$

In the limit $N \gg \lambda$, the amplitude is

$$\frac{\langle N+1^\lambda_0|\psi(0)^\dagger|N^\lambda_0\rangle}{|\langle N+1^\lambda_0|N+1^\lambda_0\rangle\langle N^\lambda_0|N^\lambda_0\rangle|^{1/2}} \approx \left(\frac{1}{L(\lambda N)^{\lambda-1}}\right)^{1/2}. \quad (21)$$

A similar result has been obtained by Kinaret *et al.*[8] who calculated the amplitude for tunneling into the edge state of a quantum dot using a chiral Luttinger liquid model. Their result is derived using the *only* assumption that the electronic Green's function at the edge of the quantum dot takes the form

$$G(x,t) = \langle \mathcal{T}_t[\psi(x,t)\psi^\dagger(0)]\rangle = \frac{\exp(ik_F x - iE_F t)a^{\lambda-1}}{(x-vt)^\lambda} \quad (22)$$

where $a \approx 1/l_B$, $1/\lambda$ is the filling factor and v is the velocity of edge excitations. The normalised overlap matrix element they obtain is

$$|\langle N_{\mathrm{dot}}|\psi(0)|N+1_{\mathrm{dot}}\rangle|^2 = a^{\lambda-1}R^{-\lambda} \quad (23)$$

where R is the radius of the dot and, in this case, N_{dot} is the number of electrons on the dot and ψ is the 1D field operator of the edge state. To compare this with our result we need to know the number N of electrons at the edge of the FQH sample, which is just the number of electrons within a distance of approximately l_B from the edge, $N = 2\pi R l_B n_e$ where n_e is the average density of electrons. This is $N = \frac{2\pi R}{\lambda l_B}$, therefore our result Eq. 21 may be rewritten with the same scaling of the microscopic length a and macroscopic length R as Eq. 23.

In summary, we have described the many-body ground and excited states of the Calogero-Sutherland model and we have performed an exact calculation of the point contact tunneling amplitude which is equivalent to the result obtained using chiral Luttinger liquid theory.

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Appendices

A. Proof of Equations 5 and 6

Let us suppose that we are dealing with a fermionic system and consider a wavefunction of the form

$$\Psi^\xi_0 = \prod_{i<j}^{N}(z_i - z_j)^\xi \prod_{k=1}^{N} z_k^{J_0} \quad (24)$$

where ξ is an odd integer, and to satisfy periodic boundary conditions, J_0 must be an integer. The most general form of Ψ includes an additional factor of symmetric polynomials, however such states always have a higher energy. First we demonstrate that $\xi = \lambda$ and then determine the value of J_0. The kinetic part of the Hamiltonian acts on Ψ yielding

$$\begin{aligned}
h_0\Psi_0^\xi &= \frac{1}{2}\sum_{l=1}^{N}\left(z_l\frac{\partial}{\partial z_l}\right)^2\prod_{i<j}^{N}(z_i-z_j)^\xi\prod_{k=1}^{N}z_k^{J_0} \\
&= \frac{1}{2}\sum_{l=1}^{N}\left(\xi\sum_{m\neq l}\frac{z_l}{z_l-z_m}+\xi^2\sum_{m,n\neq l}\frac{z_l^2}{(z_l-z_m)(z_l-z_n)}-\xi\sum_{m\neq l}\frac{z_l^2}{(z_l-z_m)^2}\right. \\
&\qquad\left.+2\xi J_0\sum_{m\neq l}\frac{z_l}{z_l-z_m}+J_0^2\right)\Psi_0^\xi.
\end{aligned} \tag{25}$$

The first and fourth terms of (25) are proportional to

$$\sum_{l\neq m}\frac{z_l}{z_l-z_m}=\frac{1}{2}\sum_{l\neq m}\frac{z_l-z_m}{z_l-z_m}=\frac{N(N-1)}{2}. \tag{26}$$

When $m = n$ the second term has the same form as the third term; otherwise it takes the form

$$\sum_{l\neq m\neq n\neq l}^{N}\frac{z_l^2}{(z_l-z_m)(z_l-z_n)} = \sum_{l\neq m\neq n\neq l}^{N}\frac{z_l^2/3+2z_l^2/3}{(z_l-z_m)(z_l-z_n)} \tag{27}$$

$$= \sum_{l\neq m\neq n\neq l}^{N}\frac{1}{3}+\frac{2z_l^2/3+z_l(z_n+z_m)/3-z_nz_m/3}{(z_l-z_m)(z_l-z_n)}. \tag{28}$$

The first term is $\frac{N(N-1)(N-2)}{3}$ and the second term is zero. The proof is by induction on N, beginning with $N = 3$ (see Ref. 1). Then we are left with

$$h_0\Psi_0^\xi = \left(\frac{\xi N(N-1)(1+2J_0)}{4}+\frac{\xi^2N(N-1)(N-2)}{6}+\frac{\xi(\xi-1)}{2}\sum_{m\neq l}\frac{z_l^2}{(z_l-z_m)^2}+\frac{NJ_0^2}{2}\right)\Psi_0^\xi \tag{29}$$

The interaction piece of the Hamiltonian is

$$h_1\Psi_0^\xi = -\lambda(\lambda-1)\sum_{j\neq l}\frac{z_jz_l}{2(z_j-z_l)^2}\Psi_0 = \lambda(\lambda-1)\sum_{j\neq l}\left(\frac{1}{4}-\frac{z_j^2}{2(z_j-z_l)^2}\right)\Psi_0^\xi \tag{30}$$

In order for Ψ_0^ξ to be a solution there must be a cancelation between the second terms of (29) and (30), therefore $\xi = \lambda$. This will hold true even if a more complicated state with additional symmetric polynomial factors is considered. Therefore

$$(h_0+h_1)\Psi_0^\lambda = \frac{N}{12}(\lambda^2(N-1)(2N-1)+6\lambda(N-1)J_0+6J_0^2)\Psi_0^\lambda \tag{31}$$

J_0 is found by minimising the eigenvalue, keeping in mind that it is constrained to be an integer, and Eq. 6 follows.

B. Proof of Equations 9 and 10

Consider the action of h on $\Psi \equiv \Psi_0\Phi$ (see Eqs. 2-5):

$$h\Psi = e_0\Psi+\frac{1}{2}\left[\sum_{j=1}^{N}2z_j^2\frac{\partial\Psi_0}{\partial z_j}\frac{\partial\Phi}{\partial z_j}+\Psi_0\left(z_j\frac{\partial}{\partial z_j}\right)^2\Phi\right]. \tag{32}$$

In the second term, $\frac{\partial \Psi_0}{\partial z_j} = \left(\frac{J_0}{z_j} + \sum_{j,i\neq j} \frac{\lambda}{z_j - z_i} \right) \Psi_0$, therefore the second term is

$$2\Psi_0 \sum_{j,i\neq j} \left(\frac{\lambda z_j^2}{z_j - z_i} + \frac{J_0 z_j}{N-1} \right) \frac{\partial \Phi}{\partial z_j} = \Psi_0 \left(\lambda \sum_{j,i\neq j}^{N} \frac{z_j^2 + z_i z_j}{z_j - z_i} \quad \left[\pm \sum_{j=1}^{N} z_j \right] \right) \frac{\partial \Phi}{\partial z_j} \tag{33}$$

where we have substituted $J_0 = \frac{-\lambda(N-1)}{2} \quad \left[\frac{\pm 1}{2}\right]$. The first term of (33) is $\tilde{h}_1$, and the third term of (32) plus the second term of (33) yield $\tilde{h}_0$.

C. Proof of Equation 13

We will derive (10) from (13). An equivalent definition of the symmetric polynomials (see Eq. 11) is

$$\Phi_{\mathbf{n}} = \sum_P \prod_{j=1}^{N} z_{P_j}^{n_j}. \tag{34}$$

Using this we may rewrite (13) as

$$\begin{aligned}
\tilde{h}_1 \Phi_{\mathbf{n}} &= \sum_{j<k}^{N} (n_j - n_k) \sum_{s=1}^{n_j - n_k - 1} \sum_P z_{P_1}^{n_1} \ldots z_{P_j}^{n_j - s} \ldots z_{P_k}^{n_k + s} \ldots z_{P_N}^{n_N} && (35) \\
&= \sum_{j<k}^{N} (n_j - n_k) \sum_P \left(\prod_{i=1}^{N} z_{P_i}^{n_i} \right) \sum_{s=0}^{n_j - n_k - 1} \left(\frac{z_{P_k}}{z_{P_j}} \right)^s && (36) \\
&= \sum_{j<k}^{N} (n_j - n_k) \sum_P \left(\prod_{i=1}^{N} z_{P_i}^{n_i} \right) \left[\left(\frac{z_{P_k}}{z_{P_j}} \right)^{n_j - n_k} \frac{z_{P_j}}{z_{P_k} - z_{P_j}} - \frac{z_{P_j}}{z_{P_k} - z_{P_j}} \right] && (37)
\end{aligned}$$

where in the last line we have computed the sum over s. Exchanging $P_k \leftrightarrow P_j$ and then exchanging $P_i \leftrightarrow i$ the first term is

$$\sum_{j<k}^{N} (n_j - n_k) \sum_P \left(\prod_{i=1}^{N} z_{P_i}^{n_i} \right) \frac{z_{P_k}}{z_{P_j} - z_{P_k}} = \sum_{j<k}^{N} \frac{z_k}{z_j - z_k} \sum_P (n_{P_j} - n_{P_k}) \prod_{i=1}^{N} z_i^{n_{P_i}} \tag{38}$$

and under the exchange $P_i \leftrightarrow i$ the second term is

$$\sum_{j<k}^{N} \frac{z_j}{z_j - z_k} \sum_P (n_{P_j} - n_{P_k}) \left(\prod_{i=1}^{N} z_i^{n_{P_i}} \right). \tag{39}$$

Summing (38) and (39) yields the action of $\tilde{h}_1$ on $\Phi_{\mathbf{n}}$ as defined by Eq. 10.

D. Proof of Equation 19

In this section we evaluate the tunneling amplitude, Eq. 19.

$$\begin{aligned}
\langle N + 1_0^\lambda | \Psi^\dagger(0) | N_0^\lambda \rangle = \frac{(N+1)^{-1/2}}{\sqrt{L}(2\pi i)^{N+1}} \oint_{|z|=1} \frac{dz_1 \ldots dz_{N+1}}{z_1 \ldots z_{N+1}} \left[\prod_{i<j}^{N+1} \left(\frac{1}{z_i} - \frac{1}{z_j} \right)^\lambda \prod_{k=1}^{N+1} z_k^{\frac{\lambda N}{2}} \right] \\
\times A_{N+1} \left\{ \prod_{m<n}^{N} (z_m - z_n)^\lambda \prod_{p=1}^{N} z_p^{-\left(\frac{\lambda(N-1)+1}{2}\right)} \sum_{q=-C^-}^{C^+} z_{N+1}^q \right\} \quad (40)
\end{aligned}$$

where we have assumed that N is even and the first choice of ground state is used in the ket. The integrand is symmetric under exchange of any $z_1 \dots z_{N+1}$, therefore we need only retain the $(N+1)$th term in the A-sum of the second factor and multiply the result by $(N+1)$:

$$= \frac{(N+1)^{1/2}}{\sqrt{L}(2\pi i)^{N+1}} \oint_{|z|=1} \frac{dz_1 \dots dz_{N+1}}{z_1 \dots z_{N+1}} \prod_{i<j}^{N} \left(\frac{1}{z_i} - \frac{1}{z_j}\right)^{\lambda} (z_i - z_j)^{\lambda} \prod_{k=1}^{N} \left(\frac{1}{z_k} - \frac{1}{z_{N+1}}\right)^{\lambda} z_k^{\frac{(\lambda)-1}{2}} \sum_{q=-C^-}^{C^+} z_{N+1}^{q+\frac{\lambda N}{2}}$$

The total power of $z_1 \dots z_N$ must vanish in the integrand, therefore only the term $q = N/2$ contributes from the sum, and the limits may be extended to $\pm\infty$ without changing the result. Then the sum is equal to $\delta(x_{N+1})$ and the integral is

$$\begin{aligned} &= \frac{(N+1)^{1/2}}{\sqrt{L}(2\pi i)^{N}} \oint_{|z|=1} \frac{dz_1 \dots dz_{N}}{z_1 \dots z_{N}} \prod_{i<j}^{N} \left(\frac{1}{z_i} - \frac{1}{z_j}\right)^{\lambda} (z_i - z_j)^{\lambda} \prod_{k=1}^{N} \left(\frac{1}{z_k} - 1\right)^{\lambda} z_k^{\frac{(\lambda-1)}{2}} \\ &= \frac{(N+1)^{1/2}}{\sqrt{L}} [1] \left\{ \left(\frac{1}{z_i} - \frac{1}{z_j}\right)^{\lambda} \prod_{i<j}^{N} (z_i - z_j)^{\lambda} \prod_{k=1}^{N} \left(\frac{1}{z_k} - 1\right)^{\lambda} z_k^{\frac{(\lambda-1)}{2}} \right\} \end{aligned} \tag{41}$$

where $[1]\{\}$ represents the coefficient of unity of the polynomial inside the parentheses. We use an exact expression given by Kadell [9] to evaluate this coefficient,

$$[1]\{\} = N! f_N^{\lambda} (-1)^{\lambda N + \frac{N(1-\lambda)}{2}} \prod_{i=1}^{N} \frac{(i\lambda)!}{\left((i-1)\lambda + \frac{\lambda-1}{2}\right)! \left(i\lambda - \frac{\lambda-1}{2}\right)!} \tag{42}$$

where

$$f_N^{\lambda} = \prod_{1 \le i < j \le N} \frac{((j-i+1)\lambda - 1)!}{((j-i)\lambda - 1)!} \tag{43}$$

$$= \left[\frac{(2\lambda-1)!}{(\lambda-1)!}\right]^{N-1} \left[\frac{(3\lambda-1)!}{(2\lambda-1)!}\right]^{N-2} \cdots \left[\frac{(N\lambda-1)!}{((N-1)\lambda-1)!}\right]^{1} \tag{44}$$

$$= \frac{1}{[(\lambda-1)!]^N} \prod_{i=1}^{N} (i\lambda - 1)! \tag{45}$$

Substituting (45) in (42) we obtain

$$[1]\{\} = \frac{N!}{[(\lambda-1)!]^N} \prod_{i=1}^{N} \prod_{j=1}^{\frac{\lambda-1}{2}} (i\lambda - j + 1)(i\lambda - j). \tag{46}$$

Now we need to find the behaviour of this function for $N \gg \lambda$. First consider the relation between the result for $N+1$ and N:

$$\frac{[1]\{\}_{N+1}}{[1]\{\}_N} = \frac{N+1}{(\lambda-1)!} \prod_{j=1}^{\frac{\lambda-1}{2}} ((N+1)\lambda - j + 1)((N+1)\lambda - j) \tag{47}$$

$$\approx \frac{N+1}{(\lambda-1)!} [\lambda(N+1)]^{\lambda-1} = \frac{[(N+1)\lambda]^{\lambda}}{\lambda!} \approx \binom{(N+1)\lambda}{\lambda} \tag{48}$$

Therefore we may approximate

$$[1]\{\} \approx \frac{(N\lambda)!}{(\lambda!)^N} = \langle N_0^{\lambda} | N_0^{\lambda} \rangle \tag{49}$$

and Eq. 19 follows. The other choice of ground state yields the same result. In general the sign is $(-1)^{\frac{N}{2}(\lambda \pm 1)}$, which is always positive for N even, while for N odd it depends on the choice of the $N+1$ particle ground state. In other words, one has the freedom to define the overall phase (*i.e.* sign) of the ground state $|N_0^{\lambda}\rangle$.

E. Jack Polynomials

In this section we discuss the relation between Jack polynomials and the basis of symmetric polynomials defined by Eq. 11 using normalisation convention of MacDonald [10]. Jack polynomials form a complete basis of symmetric polynomials and are the solutions to Eqs. 9 and 10. They are denoted by $P^\lambda_{\mathbf{n}}$ where $n = (n_1, \ldots, n_N)$, $n_1 \geq \ldots \geq n_N$. If n_N is negative one should boost the polynomial by $\prod_{k=1}^{N} z_k^{-n_N}$ in order to evaluate the normalisation[11].

The normalisation of the eigenfunctions $\Psi^\lambda_0 P^\lambda_{\mathbf{n}}$ is given by [10]

$$\begin{aligned} \langle N^\lambda_{\mathbf{n}} | N^\lambda_{\mathbf{n}} \rangle &\equiv \left(\frac{1}{2\pi i}\right)^N \oint_{|z|=1} \frac{dz_1 \ldots z_N}{z_1 \ldots z_N} \prod_{i<j}^{N} (z_i - z_j)^\lambda \left(\frac{1}{z_i} - \frac{1}{z_j}\right)^\lambda \bar{P}^\lambda_{\mathbf{n}}(z_1, \ldots z_N) P^\lambda_{\mathbf{n}}(z_1, \ldots z_N) \\ &= N! \prod_{i<j}^{N} \frac{\Gamma(\xi_i - \xi_j + \lambda)\Gamma(\xi_i - \xi_j - \lambda + 1)}{\Gamma(\xi_i - \xi_j)\Gamma(\xi_i - \xi_j + 1)} \end{aligned} \tag{50}$$

where $\xi_{\equiv} n_i - \lambda i$. This choice of scalar product differs from MacDonald's by a factor of $N!$. The normalisation of the ground state Ψ_0 (15) may be found by expanding the product

$$\langle N^\lambda_0 | N^\lambda_0 \rangle = N! \prod_{1 \leq i < j \leq N} \frac{[(j-i+1)\lambda - 1]![(j-i-1)\lambda]!}{[(j-i)\lambda - 1)![(j-i)\lambda]!}. \tag{51}$$

The relation between the eigenfunctions $P_{(1)^n}$ and $\Phi_{(1)^n}$, where $(1)^n = (\underbrace{1, \ldots, 1}_{n}, 0 \ldots, 0)$ is just

$$\Phi_{(1)^n} = n!(N-n)! P^\lambda_{(1)^n} \tag{52}$$

since, according to MacDonald, $P_{(1)^n}$ is a monomial symmetric function,

$$P_{(1)^n} = e_n \equiv \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{53}$$

where e_n is an elementary symmetric function, which is just the sum of all products of n distinct variables, and the coefficient of $\Phi_{(1)^n}$ is $n!(N-n)!$.

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