Emission of inertial waves by baroclinically unstable flows: laboratory experiments with Altimetric Imaging Velocimetry

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Abstract

Results from new experiments on baroclinic instability of a coastal jet demonstrate that this almost balanced flow spontaneously emits inertial waves when the Rossby radius of deformation is relatively small such that the characteristics of baroclinic meanders match the dispersion relation for the inertial waves. The energy of the waves is small compared to the energy of the flow. A single event of wave emission is identified in the experiment with larger radius of deformation and is interpreted in terms of vorticity dynamics. The flows are generated on a laboratory polar $\beta$-plane where the Coriolis parameter varies quadratically with latitude. A new method for imaging the rotating flows which we call “Altimetric Imaging Velocimetry” is employed. Optical color coding of slopes of the free-surface elevation field allows us to derive the fields of pressure, surface elevation, geostrophic velocity or the ‘gradient wind’ velocity with very high spatial resolution (typically several million vectors) limited largely by the pixel resolution of the available imaging sensors. The technique is particularly suited for the investigations of small-amplitude waves which are often difficult to detect by other methods.
1 Introduction

A balanced flow is the vortical predominantly non-divergent flow which can be described by an appropriate balance relation, of which semi-geostrophy and quasi-geostrophy are examples. The divergent flow in the form of inertia-gravity waves is therefore filtered out and assumed to be negligible. However, under certain circumstances, the balanced flow can undergo further adjustment toward a new (balanced) state (e.g. Ford, McIntyre and Norton 2000, McIntyre and Norton 2000). In the process of adjustment the flow radiates waves. Since the waves can propagate far away without significant decay they can be dynamically important despite the fact that their energy is relatively small. The exact circumstances through which the wave radiation takes place are yet to be established. It is also important to measure the amount of energy radiated in these events. Acoustic wave generation by turbulence in a compressible fluid is analogous, and Lighthill’s (1952) classic analysis regards compact regions of turbulence as radiating sound like a quadrupole source. Ford (1994) derived a similar source term for gravity waves in a rotating shallow water system. These important results provide a theoretical basis for analysis, however they do not specify which features of a balanced flow favor these nonlinear source terms to be large enough to cause significant wave radiation.

Laboratory experiments on spontaneous emission are relatively few due to the fact that it is technically challenging to measure globally small amplitude waves in the evolving vortical flow. In Lovegrove, Read and Richards (2000) as well as in a recent paper by Williams, Haine and Read (2005) a rotating two-layer annulus experiment is described where the lower layer contained a fluid which rotates the polarization plane of light thus providing a global method of visualization. Inertia-gravity waves were detected in the fluid where large scale baroclinic instability developed. The instability occurred because the two-layer system was driven away from its equilibrium by applying a torque by a rotating lid.
Radiation of gravity waves from a simpler vortical flow was described by Afanasyev (2003). The idea that waves are emitted as a result of interaction of vortices which are fundamental elements of geophysical turbulence was verified experimentally in a two-layer (non-rotating) fluid. Interfacial gravity waves were detected after the collision of two translating vortex dipoles. The waves were radiated when the fluid parcels underwent strong deceleration such that \( \Delta U \Delta T / \lambda = 1 \), where \( \Delta U \) is the characteristic velocity change of a parcel occurring over the characteristic time period \( \Delta T \) of the event and \( \lambda \) is the wavelength of the wave obtained from an appropriate dispersion relation. The phase speed of the wave matched the translation speed of dipoles emerging after the collision which made the Froude number equal to unity. Quadrupolar spatial distribution of vorticity seemed to be equally important for the emission to occur. Thus a combination of factors, namely large acceleration, match between the spatial and temporal parameters of the pressure disturbance due to the vortical motion and those of a wave (as specified by an appropriate dispersion relation) and, finally, the quadrupolar vorticity distribution (which is perhaps the reason for large acceleration in the first place). The energy of the wave in the experiment was estimated to be a few percent of the total energy of the flow. The measurements were performed using the optical thickness of a dyed thin interfacial layer.

A natural choice of flow for investigating the spontaneous imbalance is a baroclinic jet. A few recent numerical simulations (O’Sullivan and Dunkerton 1995, Zhang 2004, Viudez and Dritschel 2006) describe the emission of inertia-gravity waves by unstable baroclinic jet. In simulations by Viudez and Dritschel (2006) the emission is highly localized in space and occurs only "at the largest curvature side of the emerging highly ageostrophic anticyclonic vortex". Their observation of large curvature is consistent with the notion of large acceleration of fluid parcels required for the emission of waves. For our laboratory investigation we also chose a baroclinic jet as a prototype flow. The boundary jet is created on the surface of a thick heavier layer of water such that the Rossby radius of deformation is relatively small. Small-scale meanders occurred due to baroclinic instability. Baroclinically unstable coastal currents were studies in the laboratory by Griffiths and
Linden (1981). In their experiments the buoyant fluid was released uniformly along the cylindrical boundary. The wavelength and phase velocity of the developing baroclinic instability were measured and compared with a theoretical model which included a two-layer wedge-shaped fluid in a channel with frictional dissipation due to Ekman layers. They demonstrated that the current became unstable when it reached a width at which the critical value of the velocity difference between the layers was exceeded. The release of fluid from the point source (similar to that we use here) was also studied and the currents were observed to become unstable. The main control parameters of the problem are the Froude number, Ekman number and the ratio of layer depths. The characteristics of the baroclinic instability observed in our experiments are discussed in Section 4 and are consistent with those observed in Griffiths and Linden (1981).

In this paper we describe the emission of inertial waves by the baroclinic meanders. We employ a new experimental method of Altimetric Imaging Velocimetry (Afanasyev, Rhines and Lindahl 2007). This method seems to be particularly suited for the investigation of wave emission since it is a global imaging technique. The method of Optical Altimetry on which basis the Velocimetry method has been developed, was described in a recent paper by Rhines, Lindahl and Mendez 2006. The method is based on the use of the paraboloidal surface of water as a Newtonian telescope. If the surface is illuminated by a light source the slopes of the perturbations of the surface height can be visualized using the gradient of brightness across the light source. Altimetric Imaging Velocimetry (AIV) is the extension of Optical Altimetry which allows us to measure two components of the gradient of surface height. The AIV method is based on a simple idea of color coding when a gradient of color is used in two dimensions across the light source. Geostrophic (or gradient wind) balance relates gradient of pressure to the velocity field thus providing the valuable characteristics of the flow. To obtain full information about the flow a total velocity can be measured in addition to the geostrophic velocity. Using seeding particles the total velocity field can be measured by regular PIV albeit with much smaller resolution. For the flows with small Rossby number however the geostrophic velocity alone is often enough for a comprehensive analysis.
In the following sections of this paper we report the results of laboratory experiments investigating different regimes of an unstable baroclinic flow. The experimental apparatus and the details of the AIV method are described in Section 2. The visualizations of the flow as well as the results and interpretation of measurements of the flow characteristics are given in Section 3. Further analysis of the baroclinic instability of the coastal jet is offered in Section 4 while inertial waves in the interior of the tank are discussed in Section 5. Concluding remarks are offered in Section 6.

2 Experimental technique

The experiments were performed in a circular tank of diameter \( D = 97 \) cm filled initially with a layer of salt water of depth \( H_0 = 12.5 \) cm. The fluid was dyed a dark-blue color by methylene blue dye to reduce the unwanted reflection of light from the bottom of the tank. The flow was generated by injecting fresh water at the surface of the heavier layer. A peristaltic pump was used to provide a constant volume flux. The fluid was injected through a thin vertical pipe with a diffuser. The pipe was located adjacent to the wall of the tank such that a baroclinic boundary jet formed readily along the wall. The tank was installed on the rotating table and rotated in an anticlockwise direction with a rate of \( \Omega = 2.32 \) rad/s. The rotating table was driven by a servo-controlled direct drive d.c. motor such that the stability of the rotation speed was approximately 0.01%. The rotational axis was carefully aligned with gravity to avoid large scale tides in the tank. (Additional details of the rotating table characteristics and other relevant properties of the laboratory can be found on a ‘fluids Wiki’, http://gfd.ocean.washington.edu/wiki).

The free surface of the rotating fluid is a paraboloid described by

\[
\frac{z}{H_0} = \frac{\Omega^2}{2g} \left( r^2 - \frac{D^2}{8} \right),
\]

where \( z \) is the height of the free surface, \( H_0 \) is the depth of the water layer in the absence of rotation, \( g \) is gravitational acceleration and \( r \) is the radial distance from the axis of rotation in the horizontal plane. Above the water surface a color slide is located at height \( H \) (Figure 1) slightly off
the rotation axis which is also the optical axis of the paraboloid. The slide is printed out on
transparency using the following color coding. The colors are initially defined in CIE L*a*b color
space. This color space was developed by the International Commission on Illumination
(Commission Internationale d’Eclairage) as a device independent reference model in an attempt to
linearize the perceptibility of color differences. The color is described by three parameters: lightness
\( L \), and chromaticities \( a \) and \( b \). In our slide \( L \) was fixed at 75 (0 being black while 100 is
white). Parameters \( a \) and \( b \) vary from negative to positive values such that the color changes from
green to magenta in the \( x \) direction and from blue to yellow in the \( y \) direction. The L*a*b is then
converted to sRGB color space for rendering the image on a computer display or for printing. The
slide is illuminated by fluorescent lamps from behind.

The reflection of the slide in the water is observed by a video (or photo) camera which is
located at height \( H_c \) above the water. When the water in the tank is starting to spin up, one can
observe that the reflection of the slide gradually stretches over the surface of the fluid. At certain
rotation rate \( \Omega_0 \) (null point) the slide is stretched to such an extent that the entire surface of water is
illuminated by one color only. This indicates that any ray coming to the camera and reflected from
any point at the surface originates from one particular point on the slide. The reflection law allows
one to obtain the expression for the rotation rate of the fluid at the null point:

\[
\Omega_0^2 = \frac{g}{2HH_c} \left( \frac{H}{c} + H \right),
\]

where \( H, H_c \) are the heights of the slide and the camera.

The experiments are normally performed when the rotation rate of the fluid is near the null
point. Any perturbation of the free surface results in the appearance of a color different from that of
the null point. The particular values of color can be identified with their locations on the slide. For
this purpose a calibration is performed. At some rotation rate different from that of the null point
when an image of the entire slide can be rendered, the variation of parameters \( a \) and \( b \) is measured
in the \( x \) and \( y \)-directions across the slide. The dependences obtained for \( a \) and \( b \) in the calibration
are used in the experiments to map each particular color observed to a location of this color on the slide. The slope of the surface at any point can then be calculated using the following relation

$$\eta_x = \frac{X_{\text{obs}} - X}{2R},$$  \hspace{1cm} (2.3)$$

where $\eta$ is the surface elevation such that $\eta_x = \frac{\partial \eta}{\partial x}$ is the x-component of the slope, $X_{\text{obs}}$ is the location on the slide of the particular color observed, $X$ is the x-coordinate of the null point color (usually at the center of the slide), and $R$ is the distance between the point on the surface of the fluid and the slide. Here x and y are the coordinates in the horizontal plane with the origin at the center of rotation of the tank. The y-component of the slope is calculated in a similar manner. These slopes are therefore the deflections of the surface from the null point paraboloid. Geostrophic velocities and vorticity can then be easily calculated as follows:

$$\mathbf{U}_g = (u_g, v_g) = \frac{g}{f} (-\eta_x, \eta_y), \quad \zeta_g = \text{curl}(\mathbf{U}_g)$$  \hspace{1cm} (2.4)$$

where $f = 2\Omega$ is the Coriolis parameter. A higher order approximation of the velocity is provided by the gradient wind balance

$$\tilde{k} \times \mathbf{U} (\kappa \mathbf{U} + f) = -g \nabla \eta$$  \hspace{1cm} (2.5)$$

where $\tilde{k}$ is the vertical unit vector and $\kappa$ is the local curvature of the streamlines, can also be used for the purpose of calculating the velocity field. Curvature of the tangent line to the velocity vector is a scalar field given by the vertical component of the curl of the normalized velocity vector, $\kappa = \tilde{k} \cdot \text{curl}(\mathbf{U})$. The gradient wind relation includes a centripetal acceleration term $\kappa U^2$, which approximates nonlinearity for flows which are close to steady state. The relative magnitude of this term is of the order of the Rossby number. The curvature field can be estimated from the geostrophic velocity field in the first approximation. The gradient wind velocity is then found from (2.5) in the form

$$\mathbf{U} = \mathbf{U}_g + \kappa \frac{U}{f} \mathbf{U}_g$$  \hspace{1cm} (2.6)$$
Since the measurements of velocity are based on the signature of motions at the surface of fluid, it is important to show that surface tension effects are not important. Consider a perturbation of the surface of height \( \eta \) and of horizontal scale \( L \). The hydrostatic pressure due to the surface elevation is \( \rho g \eta \) while the pressure due to the surface curvature is \( \sigma / r_c \) where \( r_c \) is the radius of curvature of the surface such that \( r_c = L^2 / \eta \). The relative importance of the surface tension is then given by the ratio of pressures \( \frac{\sigma}{\rho g L^2} \). For an air-water interface the value of this ratio is \( 0.074L^{-2} \) where \( L \) is measured in centimeters. In our experiments we observe motions with scales of a few centimeters and larger which fact allows us to safely neglect the effect of the surface tension.

In a separate experiment we employed an optical thickness method to measure the depth of the injected (upper) layer. In this experiment the fresh water injected from the source was dyed a red color by food dye (FD&C Red No. 40) while the fluid in the tank was transparent. The tank was illuminated from below by fluorescent lamps with a diffuser. The map of the intensity of the red color component in the flow corresponds to the depth of the upper layer if properly calibrated. For the purpose of calibration a small cuvette with sloping bottom was placed above the surface of water near the center of the tank. This cuvette contained the same red colored water that was used for the injection. Since the depth of water varies linearly in the cuvette, it can be simply related to the color intensity.

If a standard resolution video camera is used in the experiment a digitized video frame is typically of size 720 × 480 pixels. AIV allows us to render a velocity vector at virtually every pixel of the image. The spatial resolution is therefore only limited by the size of the imager used in the experiment. At the present time, when imagers of over 3 Megapixel resolution are becoming routinely available the method provides an opportunity for laboratory experiments to compete with numerical simulations in the detailed rendering of the flow field. The ability to create a 3D flow with relatively high value of the Reynolds number (Re) can be considered an advantage of the laboratory experiment.
3 Experimental results

In the first series of experiments the fresh water was released onto the surface of a layer of depth $H_0=12.5$ cm and of density $\rho_2 = 1.028$ g cm$^{-3}$. A typical sequence of slope color maps with overlaid geostrophic velocity vectors which were calculated by AIV, is demonstrated in Figure 2. The released fluid initially forms an (unstable) anticyclonic lens around the source. At the same time a boundary jet (gravity current) forms at the wall such that the wall is to the right and the Coriolis force is perpendicular to the wall. The nose of the jet propagates in the cyclonic direction around the tank and eventually a closed cyclonic circulation is established in the tank. The baroclinic jet leaning on the wall becomes unstable and small scale meanders form along the jet. These small-scale initial instabilities can be clearly seen in Figure 2 a between 12 and 6 o'clock. Larger meanders grow near the source and then propagate with the current. It is interesting to note that the small-scale instability was observed to reappear on the large-scale meanders in the form of a chain of small vortices. One of these "baroclinic wave packets" in between the sections of larger meanders is indicated in Figure 2 d by an arrow at 2 o'clock. These vortices are especially clear in the video sequences of the experiment.

It is important to obtain information about the vertical structure of the upper layer formed by the injected fluid. For this purpose we performed a separate experiment with the same control parameters as those used in the AIV experiment. The optical thickness method was employed to obtain the map of the depth of the upper layer. Panel (f) in figure 2 shows the contour plot of depth varying from 0 to 3 cm in the flow with fully developed meanders. The contours are overlaid on the image of the flow visualized by red color. The image shows that the upper layer formed by the flow injected from the source does not cover the interior of the tank but rather forms a wedge shaped layer (in cross section) at the wall.

The visual analysis of video sequences also revealed that very low amplitude and small scale waves propagate in the unstratified interior of the tank. Characteristic propagation of phase in the
radial direction from the center of the tank toward the walls indicates that these are non-hydrostatic inertial waves where the energy propagates in the opposite direction to the phase. These waves propagate in the homogeneous density layer and are emitted by baroclinic perturbations of the jet at the wall of the tank. These waves can be seen in plates b (where they are indicated by an arrow), c and e (zoomed image) in Figure 2 although it is somewhat challenging to notice them on still images even after digital enhancement of contrast.

In order to visualize the waves more clearly and to measure the major characteristics of the waves a Hovmöller plot was rendered (Figure 3). This plot is a space-time diagram which shows the temporal variation of the x-component of the gradient wind velocity along the y-axis across the tank. In the diagram the displayed range of the variation of the velocity was intentionally narrowed down such that the small amplitude motions were visible. The upper part of the plot shows the flow at the point at 12 o'clock, while the lower part shows the point at 6 o'clock. The passing of small and large meanders through these points can be seen clearly in the diagram. The distinct crests and troughs of inertial waves can also be seen clearly in the interior region of the diagram as a succession of lines. The Hovmöller diagram allows one to point out the origins of the wave packets, the waves are being emitted by small scale meanders only. There is also an indication that larger meanders generate Rossby waves which are seen as almost horizontal white bands originating from these meanders. Crests and troughs of the Rossby wave are perhaps better visible at Figure 2 c and d as large alternating bands in the interior of the domain at approximately 7 - 8 o'clock. Rossby waves generated on the polar $\beta$ -plane and observed by the AIV method are discussed in more detail in Afanasyev, Rhines and Lindahl (2007).

The velocity profiles along the same line that is used for the Hovmöller plot are demonstrated in Figure 4. These profiles show the x-component of velocity in the baroclinic jet at the point at 12 o’clock as well as in the adjacent area in the interior of the tank. The profiles are measured with the time interval of 0.5 s. The crests of the wave (marked by arrows) in the interior propagate towards the jet which implies the energy propagating in the opposite direction. At the same time the flank of
the jet decelerates. Although the geostrophic (or gradient wind) velocity profiles show the spatial pattern of the wave as well as its propagation, a different velocity inversion is required to obtain the correct values of the velocity components in the wave. A simplified equation (which follows from equation (5.8) in Section 5) relating the surface slopes and the surface velocity of the inertial wave can be used for this purpose

\[ (u, v) = \frac{g}{f(1 - \omega'^2)} (-\eta_x, \eta_y, \eta_x' + i\omega' \eta_y) \]  

(3.1)

Here \( \eta \) is the surface elevation and \( \omega' = \omega/f \) is the dimensionless frequency. More detailed consideration of the relations between the characteristics of the wave and its signature on the surface is offered in Section 5. Here we use (3.1) to obtain the profile of the x-component of the velocity in the unstratified interior of the tank where the flow is identified as the inertial wave. Altimetry gives the surface slopes while the frequency of the wave can be measured separately from the video sequences of the flow. Note that for small \( \omega' \) the surface velocity in the wave reduces to the geostrophic velocity. The profile obtained as a result of this inversion is shown in Figure 4. Since in this experiment the dimensionless frequency is relatively small \( \omega' = 0.23 \) the surface velocity in the wave does not differ significantly from the geostrophic velocity.

An alternative way to visualize the baroclinic flow and the inertial waves is to plot a map of the curvature \( \kappa = \vec{k} \cdot \text{curl}(\vec{U}/U) \) of the instantaneous streamlines. Curvature is different from the vorticity because its definition only uses the information about the direction of the velocity vector rather than its magnitude. Curvature maps are shown in Figure 5 for two instances when the small meanders prevailed and the wave emission was relatively strong as well as for a later time when a highly ordered chain of larger meanders were formed and the wave emission was not so pronounced. The vorticity field is shown for comparison with curvature in the same Figure in panels c and d. The flow field at the same instances is visualized also by the topography of the surface elevation in Figure 6. The amplitude of the elevation change is approximately 0.2 cm in this experiment. The isolines of the surface topography coincide with the instantaneous geostrophic
streamlines. The curvature field has extremes in the locations of the flow where the fluid parcels make sharp turns. These extremes are much localized in space and show critical points in the flow including centers of vortices as well as the flanks of the jets. Figure 5 a shows that radiated waves are the extension of the small-scale meanders (jets) protruding in the radial direction. The topology of the flow in plate b is different to a certain extent, the jets form a more closed structure oriented mainly in the azimuthal direction.

4 Baroclinic instability

It is instructive to consider in more detail the dimensional as well as major non-dimensional parameters of both waves and the baroclinic flow. Comparison of these characteristics with known theoretical results provides a consistency check and clarifies the regime of the flow. Let us first consider the unstable baroclinic jet. A simple volumetric estimate allows us to obtain the mean depth of the coastal current \( h_i = 2 \) cm. The total volume of fluid released from the source, the average width of the jet and its length (the circumference of the tank) as well as the wedge-like form of the flow were taken into account. This estimate is consistent with the direct measurements of the depth by optical thickness method (Figure 2 f). An important control parameter for the baroclinic instability is then the ratio of layer depths \( \gamma = h_i/(D-h_i) = 0.2 \). A Froude number can be introduced as \( F = f^2 W^2 / g h_i \), where \( f = 2\Omega = 4.6 \) rad/s is the Coriolis parameter, \( g' = 28 \) cms\(^{-2}\) is the reduced gravity and \( W = 5 \) cm is the width of the current (estimated by the lateral extent of the colored fluid in optical thickness experiment). This gives the value \( F \approx 10 \). According to the linear theory of Griffiths & Linden (1981) the critical value of the Froude number is approximately 3 for \( \gamma = 0.2 \). Our current is therefore supercritical. The Rossby radius of deformation \( R_d = \sqrt{gh_i/f} = 1.5 \) cm can also be used as a dimensional control parameter for this flow. The wavelength, \( \lambda_{\text{inst}} \) of small meanders measured from the experimental images is approximately 5.5 cm such that \( \lambda_{\text{inst}} = 3.5 R_d \) which is a typical ratio for baroclinic instability (e.g. Blokhina & Afanasyev 2003).
The frequency with which the baroclinic meanders are passing through can also be estimated from the Hovmöller plot to be $\omega_{\text{baro}}/f = 0.3$ for small meanders and 0.07 for large meanders. Here the circular frequency is nondimensionalized by the Coriolis parameter. The extent to which the flow can be considered geostrophic is characterized by the smallness of the Rossby number of the flow. Here we estimated the maximum values of the Rossby number by measuring the peak values of the magnitude of the geostrophic vorticity $\zeta$ in the jet such that $Ro = \zeta f$. The values $Ro \leq 0.2$ were obtained except for the beginning of the experiment in the vicinity of the source where the Rossby number is of order of unity.

Useful information about the dynamics of the baroclinic flow can also be obtained from the investigation of its spectral characteristics. The procedure that we employed is explained in more detail in Afanasyev and Wells (2005). Briefly, the Cartesian velocity components measured on the Cartesian grid were interpolated into polar grid and transformed into radial and azimuthal velocity components. Each velocity component was decomposed using a Fourier transform in $\theta$ and Bessel function decomposition in $r$ such that

$$v(r, \theta) = \sum_m \sum_j v_{mj} \exp(im\theta)J_m(\alpha_{mj}) \frac{2r}{D}. \tag{4.1}$$

where $\alpha_{mj}$ is zeroes of the Bessel function. The two-dimensional power spectrum $E_{mj} = v_{mj}^*v_{mj}$ in wavenumber space $(m, j)$ was then calculated. A typical spectrum is shown in Figure 7 for $t = 226$ s. The zonal energy is represented by the first line $(m = 0)$ of the array $E_{mj}$. It is clear that a significant amount of energy is concentrated in the zonal component of the flow. There is also a distinct peak in the spectral distribution at approximately $m = 23$ and $j = 9$. This peak gives the characteristic spatial frequency of the baroclinic instability (meanders) of the boundary jet. The signature of inertial waves in the spectrum can be expected in the region close to that of the baroclinic meanders. However, the waves cannot be easily identified and separated from other motions in this spectrum.
5 Inertial waves

Consider next the inertial waves in the interior of the tank. Since we investigate the flows by their signature on the surface it is important to establish the appropriate relations for the inertial waves. Starting with linearized equations of motion for a homogeneous density layer of depth $H_0$

$$\begin{align*}
\partial_t u - fv &= -\frac{1}{\rho} \partial_z p \\
\partial_t v + fu &= -\frac{1}{\rho} \partial_y p \\
\partial_t w &= -\frac{1}{\rho} \partial_x p \\
\partial_x u + \partial_y v + \partial_z w &= 0
\end{align*}$$

with boundary conditions

$$\begin{align*}
w(0) &= \partial_z \eta \\
p(0) &= \rho g \eta \\
w(-H_0) &= 0
\end{align*}$$

we look for the solution in the form of horizontally propagating harmonics with vertical structure determined by their $z$-dependent amplitudes

$$(u, v, w, p) = (u_0(z), v_0(z), w_0(z), p_0(z), \eta_0) \exp(i(\omega x - kx - ly)).$$

A single equation for pressure is then obtained

$$\frac{\partial^2 p_0}{\partial z^2} + \frac{\omega^2 (k^2 + l^2)}{1 - \omega^2} p_0 = 0$$

with boundary conditions $\partial_z p_0(-H_0) = 0$ and $\partial_z p_0(0) = \frac{\omega f}{g} p_0(0)$. Here $\omega' = \omega f$ is the dimensionless frequency. The solution of (5.4) is given by

$$p_0 = \frac{\rho g \eta_0}{\cos \gamma_n H_0} \cos \gamma_n(z + H_0)$$

with dispersion relation

$$-\gamma_n \tan \gamma_n H_0 = \omega'^2 - \frac{f^2}{g},$$
where
\[
\gamma_n = \sqrt{\frac{\omega^2 (k^2 + l^2)}{1 - \omega^2}}. \quad (5.7)
\]

Velocity components can then be easily obtained in the following form:
\[
\begin{align*}
\quad u &= \frac{-g\eta_0}{f} \frac{\cos \gamma_n (z + H_0)}{(1 - \omega^2) \cos \gamma_n H_0} \left[ l \sin (\omega' ft - kx - ly) + k \omega' \cos (\omega' ft - kx - ly) \right] \\
\quad v &= \frac{g\eta_0}{f} \frac{\cos \gamma_n (z + H_0)}{(1 - \omega^2) \cos \gamma_n H_0} \left[ k \sin (\omega' ft - kx - ly) - l \omega' \cos (\omega' ft - kx - ly) \right] \\
\quad w &= \frac{-g\eta_0}{f} \frac{\sin \gamma_n (z + H_0)}{(1 - \omega^2) \cos \gamma_n H_0} \left[ k^2 + l^2 \right]^{1/2} \sin (\omega' ft - kx - ly)
\end{align*}
\]
\[
(5.8)
\]

The relations (5.5) and (5.8) allow us to obtain all of the characteristics of the inertial wave by its signature on the surface determined by the surface elevation \( \eta \).

To identify the particular modes that we observe in the experiments the wave number and frequency of the observed waves can be compared with the values obtained from the dispersion relation (5.6). These parameters were measured using the video sequences of the flow for a few different time periods when the waves were most distinct. The results of these measurements are given in Figure 8 where the wave number \( k \) is nondimensionalized by the barotropic radius of deformation \( R_g = \sqrt{gH_0/f} \). Solid lines show the solution of the dispersion relation for modes \( n = 1, 2 \) and 3. The comparison demonstrates that the lowest frequency mode 1 is observed in the experiments. The phase speed of the waves \( c = \omega/k = 1 \pm 0.2 \) cm/s, calculated using the measured values of frequency and wave number is in good agreement with the value of \( c \) that is measured as the slope of the lines of constant phase in the Hovmöller plot in Figure 3.

Finally, an important characteristic that needs to be measured in this experiment is the amount of energy of the wave relative to the energy of the main flow. For this purpose the kinetic energy of the flow is integrated along the y-axis from the center of the tank to the wall. In the velocity profiles along this line, the jet near the wall can be easily identified by the large values of velocity while the oscillatory part in the interior represents the wave. The kinetic energy \( E = (u_g^2 + v_g^2)/2 \) of the main flow including the jet at the wall and slow cyclonic rotation in the interior of the tank can be easily
estimated using the geostrophic velocity \((u_g, v_g)\) obtained by AIV. After the integration along the y-axis the kinetic energy is also integrated over the depth (multiplied by \(H_0\) in the interior or by the average depth of the jet near the wall). To estimate the kinetic energy \(E = (u^2 + v^2 + w^2)/2\) of the inertial waves we cannot simply use the geostrophic velocity but rather have to employ the theoretical relations (5.8). We first obtain the depth averaged squared amplitudes of the velocity components in the following form:

\[
\begin{align*}
    u_{\text{avg}}^2 & = u_g^2 \frac{1}{(1 - \omega^2)^2} \left[ \frac{1}{2 \cos^2 \gamma \gamma H_0} + \frac{\sin \gamma H_0}{2 \gamma H_0 \cos \gamma H_0} \right] \\
    v_{\text{avg}}^2 & = u_g^2 \frac{\omega^2}{(1 - \omega^2)^2} \left[ \frac{1}{2 \cos^2 \gamma \gamma H_0} + \frac{\sin \gamma H_0}{2 \gamma H_0 \cos \gamma H_0} \right] \\
    w_{\text{avg}}^2 & = u_g^2 \frac{1}{(1 - \omega^2)^2} \left[ \frac{1}{2 \cos^2 \gamma \gamma H_0} - \frac{\sin \gamma H_0}{2 \gamma H_0 \cos \gamma H_0} \right]
\end{align*}
\] (5.9)

where \(u_g = \frac{g \eta_0}{f}\) is effectively the amplitude of the geostrophic velocity measured by AIV. Here we assumed that the wave propagates in the y-direction such that the x-component of the wave vector is \(k = 0\). Typical value of the velocity amplitude was \(u_g = 0.1\) cm/s while the characteristic frequency and the wave number of the wave were such that \(\omega' = 0.2\) and \(\gamma_n H_0 = 3.4\). This gives \(u_{\text{avg}}^2, w_{\text{avg}}^2 \sim u_g^2\) while \(v_{\text{avg}}^2\) is relatively small. The kinetic energy of the wave can then be easily estimated by averaging over 3-4 wavelengths typically observed along the line in the interior of the tank and by multiplying by the depth of the layer \(H_0\). The ratio of the kinetic energy of the inertial wave and the kinetic energy of the geostrophic flow is estimated to be \(E_w/E = 0.5\%\). This is indeed a very small fraction of the total energy which supports the notion that the flow is close to being geostrophically balanced.

It is interesting to investigate further what kind of spatial distribution of vorticity in the flow is more likely to emit waves. In experiments where the radius of deformation was larger than that in the previous case, the emission of waves was not so persistent. In fact, for such experiments single emission events can be identified. Figure 9 shows a vorticity map which illustrates such an event.
The image shows the vorticity calculated from the gradient wind velocity field measured by AIV for the experiment which is identical to that discussed previously but with $R_d = 2.5$ cm. Since the radius of deformation is larger, the baroclinic meanders are larger. The meanders do not emit waves most of the time but singular emission events can be observed. The Hovmöller plot diagram in Figure 10 shows such an event. The diagram is constructed from the lines along the x-axis across the tank. These lines were cut from color frames of the original video of the experiment and then converted to grayscale. The lower part of the diagram shows the right hand side of the tank which is also shown in Figure 9. The emission occurs at approximately 30 s and the succession of lighter and darker bands can be seen in the lower part of the diagram in the time interval between 30 and 75 s. Note that there is no emission before or after this event. The amplitude of the waves is so small that they are just slightly above the background noise level. For this reason it is difficult to detect them in still images of the flow. However, these waves can more easily be tracked in the video of the experiment when they are seen in motion. Moreover, the wave packet can be connected to the evolution of a certain meander. The direction of wave emission from this meander is indicated by a white arrow in Figure 9. The growing main meander of the baroclinic jet interacts with the meander ahead such that a part of the meander is getting squeezed out forming relatively thin streaks of vorticity. We believe that this causes the emission (albeit very weak) of the inertial waves. Note that the curvature of the meander is large which is consistent with the observations by Viudez and Dritschel (2006) in their numerical simulations. The interaction (collision) of the meanders creates a complex multilayered vorticity distribution. The direction of momentum of jets (dipoles) formed by patches of oppositely signed vorticity is shown schematically by black arrows in Figure 9. Head-on arrows indicate a collision of two dipoles which constitutes a vortex quadrupole; parallel arrows pointing in the opposite directions constitute a “rotating quadrupole” (Voropayev and Afanasyev 1992, 1994). Thus a quadrupolar component is certainly present in this vorticity distribution. This fact is in agreement with the notion that quadrupolar distribution of vorticity is associated with large acceleration/decelerations of parcels and is ultimately favorable to
the emission of waves.

Concluding remarks

The results of the experiments demonstrated the radiation of inertial waves by an unstable baroclinic jet which was observed previously in high-resolution numerical simulations. The coastal jet spontaneously emits inertial waves when the Rossby radius of deformation is relatively small such that the characteristics of baroclinic meanders match the dispersion relation for the inertial waves. However, the energy of the waves is extremely small compared to the mean energy of the flow. A single event of wave emission was identified in the experiment with larger radius of deformation where the emission was relatively rare. The vorticity distribution in the meander where the emission was observed, can be interpreted to have a quadrupolar component which we believe to be favorable for the emission to occur.

The AIV technique allowed us to measure the velocity fields of the flow with high spatial resolution comparable to that of typical numerical simulations. The technique provides a tool for further use of the laboratory tank as an “analogue” computer for fluid dynamical problems.

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References


Figure captions

Figure 1. Sketch of the experimental set-up.

Figure 2. Sequence of images showing a flow produced by a source adjacent to the wall (at 9 o’clock) and visualized by color coding of slopes. Gradient wind velocity field (vectors) is calculated by AIV (every 100\textsuperscript{th} velocity vector is plotted only). The images were taken at 150 (a), 290 (b), 340 (c) and 410 s (d) after the source had been activated. Zoomed AIV image in plate (e) shows the inertial waves in the interior of the tank. Plate (f) shows the contours of depth (from 0.8 to 3 cm with the interval of 0.4 cm) superposed on the image of the flow obtained in the optical thickness experiment.

Figure 3. Hovmöller plot showing the time line of the x-component of gradient wind velocity along the y-axis across the tank.

Figure 4. Profiles (1-3) of the x-component of the gradient wind velocity along the y-axis in the vicinity of the point at 12 o’clock. First profile is measured at $t = 286$ s; the subsequent profiles are measured with the interval of 0.5 s. The distance along the y-axis is measured from the wall of the tank. Thick dashed line shows the surface velocity in the inertial wave obtained from (3.1). Comparison with profile 1 indicates that these profiles differ by a factor of $O(1)$.

Figure 5. Curvature and vorticity maps of the flow at $t = 150$ (a, c) and 590 s (b, d). Curvature (panels a and b) varies from -1 to 1 cm\textsuperscript{-1} in the color map showing the locations where fluid parcels perform cyclonic rotation in black and anticyclonic rotation in white color. The rectangular shape in the centre of the tank is an artifact which is due to the unwanted reflection of the slide from the bottom of the tank. A few bright white spots in the interior are small particles floating on the
surface. Vorticity (panels c and d) normalized by the Coriolis parameter varies from -0.3 to 0.3.

Figure 6. Topography of the surface elevation at $t = 150$ (a) and 590 s (b). The isolines are superimposed on the grayscale images of the flow. The amplitude of the elevation change is approximately 0.2 cm. The isolines of topography coincide with the instantaneous geostrophic streamlines.

Figure 7. Two-dimensional energy spectrum at $t = 226$ s. Color scale shows $\ln(E_{mj})$.

Figure 8. Frequency vs wavenumber for the inertial waves. Circles show the values measured in the experiment while solid lines show the solution of the dispersion relation (5.6) for modes $n = 1$, 2 and 3.

Figure 9. Vorticity map in the experiment with larger radius of deformation at $t = 35$ s (time origin is arbitrary and corresponds to that in Figure 10). Vorticity is normalized by the Coriolis parameter; black color (negative vorticity) shows counterclockwise (cyclonic) rotation while white color shows the areas where the local rotation is clockwise (anticyclonic). White arrow indicates the direction of radiation of inertial waves which are not visible in this diagram. Black arrows show the direction of momentum associated with the oppositely signed vorticity patches. Head-on arrows constitute a vortex quadrupole; parallel arrows pointing in the opposite directions constitute a “rotating quadrupole”.

Figure 10. Hovmöller plot showing the stack of lines cut along the x-axis across the tank from the video frames of the flow. Original color frames are converted to grayscale. The emission event can be seen in the lower part of the diagram which corresponds to the right hand side of the tank. In the upper part of the diagram (left hand side of the tank) a very high frequency perturbations are visible.
These perturbations are believed to be the Kelvin-Helmholtz rolls at the interface. They do not however cause any noticeable emission of inertial waves.
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